Reachability Analysis of Closed-Loop Switching Power Converters

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***Abstract*—A design verification method for closed-loop switching power converters is presented in this paper. The method computes the set of reachable states from an initial set of states. Case studies are presented for closed-loop buck converters using this approach. The buck converter is first modeled as a switched linear system. Two controllers are studied, first a simple hysteresis controller, and then a linear controller. The analysis method is automated and uses the hybrid systems reachability analysis tool SpaceEx. The applications and limitations of the analysis method are explored in this study.**

***Index Terms*—hybrid systems, verification, buck converter**

I. Introduction

The design and validation of switched-mode power converters typically involves numerical simulations. A variety of software tools exist for such modeling and simulation, including Simulink/Stateflow, LabView, Plexim PLECS, and PSpice. Such analysis is indispensable during the design process, as it aids the designer by giving them a first-pass view of whether the converter operates as expected. The *"expected operation"* may be based on the designer's experience and intuition, or prescribed by design specifications for input/output currents and voltages, operating temperature ranges, expected manufacturing variations in components, etc.

However, while simulations aid the designer in such first-pass analysis, they are inherently *incomplete*, in the sense that one simulation run corresponds to a single execution of the system. That is, such analysis can at best provide a counterexample that the system does not behave correctly, but cannot prove that every execution of the system operates according to the specification (due to an infinite number of possible initial conditions, component variations taking values in the reals, etc.). Additionally, while some of these tools have the capability to model the converter controller as software (e.g., Simulink/Stateflow or LabView), they generally do not do so, and tools like PSpice provide only circuit-level simulations and have no efficient capability to analyze the way the controller will actually be implemented in a modern system—via software running on a digital computer.

This paper describes a general reachability-based method for verifying closed-loop systems, applied in particular to switching power converters. We model the converters and controllers as switched linear systems, and compute an overapproximation of the set of reachable states of the closed-loop system, which are any states that may be visited by following the dynamics of the system from any initial condition (of which there may be uncountably many). The difference between reachability analysis and simulation is that reachability overapproximates all possible executions of the system, whereas simulation would model one, which due to numerical inaccuracies (lack of soundness), may not even correspond to an actual execution of the system. Thus, if reachability is sound, in the sense that if the reachable states (or overapproximations thereof) do not violate a property, then the system does not violate the property.

We use the hybrid systems [1], [2] verification tool SpaceEx for computing the reachable states [3], although there are a variety of tools that could be used [4] and have similar modeling frameworks. The limitations here are that reachability computations are expensive compared to simulations, and that the analysis is model-based and thus subject to any imperfections of the model. Nonetheless, reachability analysis allows for a more thorough, *complete* verification of a system since simulations can never capture all possible executions. A reachability method for switched-mode power converters, which relies on the ellipsoidal toolbox [5], was presented in [6]. Another reachability method using SpaceEx was applied to open-loop verification of buck converters and multi-level converters in [7]. This paper will extend on [7] by exploring the verification of closed-loop configurations of buck converters using SpaceEx.

In the following section, background on the SpaceEx architecture and underlying algorithm is presented. Section III describes the derivation of the model for a closed-loop buck converter with a linear controller. In Section IV, the linear controller model and a test hysteresis controller model are implemented in SpaceEx and explained. Conclusions and future work are discussed in Section V.

II. SpaceEx

SpaceEx is a verification platform for hybrid systems. Given a mathematical model of a hybrid system, SpaceEx ensures beyond reasonable doubt that the system satisfies some desired properties. Essentially, it is used to compute the sets of reachable states of the system. It is not just a single tool, but a development platform on which many different verification algorithms are implemented. It supports multiple methods for computing reachable sets for hybrid systems, such as PHAVer and and a variant of the Le Guernic Girard algorithm [8], [9]. The goal of SpaceEx is to enable the implementation of various methods for computing the set of reachable states using the procedure described above, as well as enabling their eventual combination and further improvements. SpaceEx is composed of a model editor, analysis core, and a web interface. It is browser based and accesses the core through a web server that can be running remotely or locally on a virtual machine.

Its reachability algorithms operate on symbolic states, which is the Cartesian product of a set of discrete states (locations) and continuous states (variable valuations). Since reachability for hybrid automata is undecidable and not guaranteed to terminate in general, a few options are available to control the algorithm. These include setting a number of maximum iterations and relative and absolute errors [10]. The set of states encountered during computation are characterized by a passed/waiting list (PWL) where the passed list is comprised of the symbolic states that have been encountered so far and the wait list contains those whose successors still have to be computed. The symbolic states of the wait list are implemented as a set of references to elements of the passed list.

The basic procedure involves first initializing the PWL and choosing a symbolic state from the list. A discrete-post is applied (possibly generating more than one state) and, subsequently, a continuous-post is applied to every generated symbolic state. The states already on the passed list are discarded and the remaining are added to the PWL, which is compressed by removing redundant states. The order the symbolic states are dropped off the wait list determines the order of computation. If the wait list is not empty, the process loops and begins again [11].

III. Closed-Loop Buck Converter Model

In this section, the derivation of the closed-loop buck converter model is discussed. A buck converter is a switched-mode, step-down DC to DC converter that is comprised of two switches (typically a transistor and a diode), an inductor, and a capacitor, as shown in Figure 1. The switches alternate between connecting the inductor to source voltage to store energy in the inductor and disconnecting the inductor and discharging into the load. In continuous conduction mode, the input voltage and the duty cycle (the period of time in a switching cycle during which the active switch conducts) determine the output voltage [12]. In an open-loop configuration, the switching frequency and duty cycle are fixed, but, in a closed-loop system, are variable (depending on control strategy) . In this particular study, the closed-loop buck converter is of primary concern, as we previously analyzed open-loop configurations [7].

The buck circuit, in continuous conduction, has two modes: one when the switch (transistor) is open and the inductor is discharging and the other when the switch is closed, with the inductor charging [12]. To begin the derivation of the closed-loop buck converter system, it is useful to first study how the open-loop system is modeled. The circuit can be modeled as a switched linear (affine) system of the form:

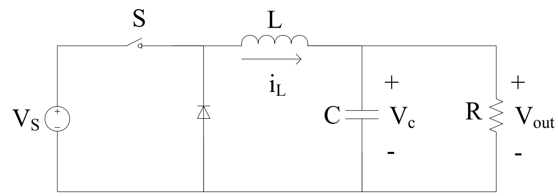

Fig. 1. Buck converter circuit.

$$\dot{x}_{\sigma(t)} = A_{\sigma(t)}x + B_{\sigma(t)},$$

where $\sigma(t) : \mathbb{R} \to \mathcal{M}$ and $\mathcal{M} = \{o, c\}$ is a function mapping time to either open-switch (o) or closed-switch mode (c) for each $i \in \mathcal{M}$, $A_i \in \mathbb{R}^{n\times n}$, and $B_i \in \mathbb{R}^n$. The capacitor voltage, V_c and the inductor current i_L are state variables of the system,

$$x = \begin{bmatrix} i_L \\ V_c \end{bmatrix}. \quad (1)$$

For both modes, the circuit system matrix can be modeled as follows:

$$A_o = A_c = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}. \quad (2)$$

where the A_o matrix is the circuit when the switch is open and A_c matrix is the circuit when the switch is closed. However, the affine input term is different for the two modes. For the closed switch, the presence of the source voltage must be accounted for:

$$B_c = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} V_s. \quad (3)$$

Conversely, for the open switch mode, the source voltage is not connected and results in the zero vector:

$$B_o = \begin{bmatrix} 0 \\ 0 \end{bmatrix} V_s. \quad (4)$$

With feedback control, the converter output is measured and the duty cycle is subsequently modulated to regulate an output variable (typically the output voltage) [13]. Typical methods for controller design are based on pole placement in the frequency domain and allow for more accurate results than in an open-loop configuration. Therefore, a stabilizing controller in the frequency domain was designed using pole placement. The controller design was adopted from Matlab/Simulink switched-mode power converter models by COPEC [14]. The equivalent linear system controller state-space components are:

$$A_{ctrl} = \begin{bmatrix} -\frac{1}{p_1} & 0 & 0 \\ -\frac{p_2}{p_1 p_3} + \frac{1}{p_3} & -\frac{1}{p_3} & 0 \\ -\frac{p_2 p_4}{p_1 p_3 p_5} + \frac{p4}{p3p5} & \frac{-p4}{p_3 p_5} + \frac{1}{p_5} & 0 \end{bmatrix}, \quad (5)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\; B_{ctrl} = \begin{bmatrix} \frac{1}{p_1} \\ \frac{p_2}{p_1 p_3} \\ \frac{p_4 p_2}{p_1 p_3 p_5} \end{bmatrix},$$

where each p_i is a real constant chosen such that the controller is stabilizing. Now, the feedback system is described as two interconnected linear systems, one of the plant—i.e., the buck converter—and one of the controller. The plant has two states, and the controller has three states. These two systems are linked by an error term, e, which is the difference between the reference voltage, V_{ref}, and output, V_{out}, voltages. That is, $e = V_{ref} - V_{out}$ and $V_{out} = V_c$, therefore $\dot{e} = -\dot{V}_c$. This error term must be factored into the model, as the converter adjusts its duty cycle according to the error value. The composed model is:

$$\dot{x} = A_{comp} x_c + B_{comp} \left(V_{ref} - V_{out} \right) \tag{6}$$

where B_{comp} is either B_c or B_o and $A_{comp} = A_c = A_o$.

After algebraic simplification, the final composed switched affine system modeling the closed-loop buck controller with the plant, controller, and error term has five states and two modes. The system is: $A_{comp} =$

$$\begin{bmatrix} 0 & \frac{-1}{L} & 0 & 0 & 0 & 0 \\ \frac{1}{C} & \frac{-1}{RC} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{p_1} & \frac{-1}{p_1} & 0 & 0 \\ 0 & 0 & \frac{-p_2}{p_1 p_3} & \frac{-p_2}{p_1 p_3} + \frac{1}{p_3} & \frac{-1}{p_3} & 0 \\ 0 & 0 & \frac{-p_4 p_2}{p_1 p_3 p_5} & \frac{-p_2 p_4}{p_1 p_3 p_5} + \frac{p_4}{p_3 p_5} & \frac{-p_4}{p_3 p_5} + \frac{1}{p_5} & 0 \end{bmatrix},$$

$$B_c = \begin{bmatrix} \frac{V_s}{L} \\ 0 \\ \frac{1}{p_1} V_{ref} \\ \frac{p_2}{p_1 p_3} V_{ref} \\ \frac{p_4 p_2}{p_1 p_3 p_5} V_{ref} \end{bmatrix},\; B_o = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{p_1} V_{ref} \\ \frac{p_2}{p_1 p_3} V_{ref} \\ \frac{p_4 p_2}{p_1 p_3 p_5} V_{ref} \end{bmatrix},$$

$$\dot{x}_{comp} = \begin{bmatrix} \dot{i}_L \\ \dot{V}_c \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}. \tag{7}$$

The controller stabilizes the plant by periodically switching between the open and closed modes based on the value of the controller state in relation to the reference voltage (i.e., it determines the duty-cycle of pulse-width modulation). This system was tested in Simulink with an input voltage of $5V$ and a reference of $2V$, and was observed to operated as expected. The result is shown in Figure 3, and the capacitor voltage stabilizes around $2V$, as illustrated in the second plot.

All other parameters of the composed system functioned as expected as well, the term "composed" referring to the combined plant and controller model. The inductor current and controller states reach steady-state. Thus, the composition of the buck converter and linear controller system appears correct.

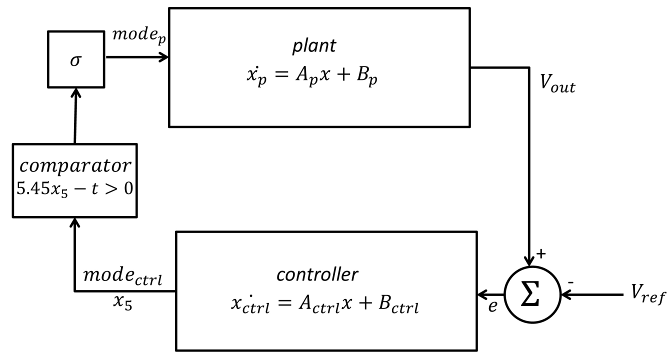

Fig. 2. Basic block diagram of buck converter and linear controller.

IV. SpaceEx Analysis

A. Hysteresis Controller

To first test closed-loop modeling capability of SpaceEx, a hysteresis controller was implemented with the buck converter model. This type of controller is a self-oscillating feedback controller that switches abruptly between two states [13]. Effectively, a control is restricted to be between a lower and an upper bound. In this case, the two states are closed-switch (charging) and open-switch (discharging) and the capacitor voltage, V_c, is controlled between bounds $V_{ref} - \delta$ and $V_{ref} + \delta$, where δ is a predetermined constant. A hybrid automaton [1] model is shown in Figure 4.

This simplified closed-loop buck converter system was modeled in SpaceEx with $\delta = 0.005$ and the following results were achieved in Figure 5. After receiving an input voltage of 12V, the capacitor voltage eventually settles down to a value around 5V. The inductor current also begins to stabilize, as seen in Figure 6. SpaceEx computes an overapproximation of the set of reachable states of the plant and controller models, which are dependent on the dynamics of the system from specific initial states. The system was initialized at $V_c = 0$V, $i_L = 0$A, and $V_s = 12$V. The system is set to be in the charging (switch-closed) mode in its initial state. This reachability analysis shows that the capacitor voltage remains within reasonable bounds around 5V after startup, which is the expected behavior of the circuit. The inductor current also stabilizes within reasonable bounds. Compared to a traditional simulation, all possible executions were overapproximated, not just one in particular as a simulation study would have illustrated. These results indicate that a closed-loop buck converter with a hysteresis controller can be effectively modeled and analyzed using hybrid systems reachability tools like SpaceEx.

B. Linear Controller

In spite of the success analyzing the open-loop buck converter and closed-loop buck converter with a hysteresis controller, our ultimate goal was to analyze a realistic closed-loop linear controller, since the hysteresis controller is not a standard controller for a buck converter. However, we started

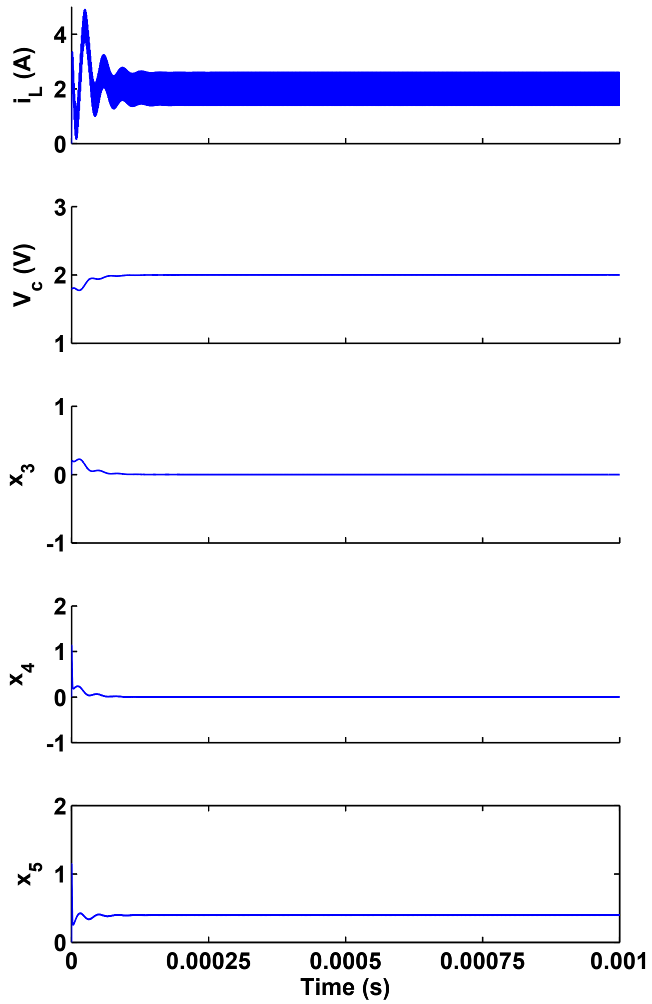

Fig. 3. Simulink simulation state variable values versus time for the closed-loop buck converter with linear controller.

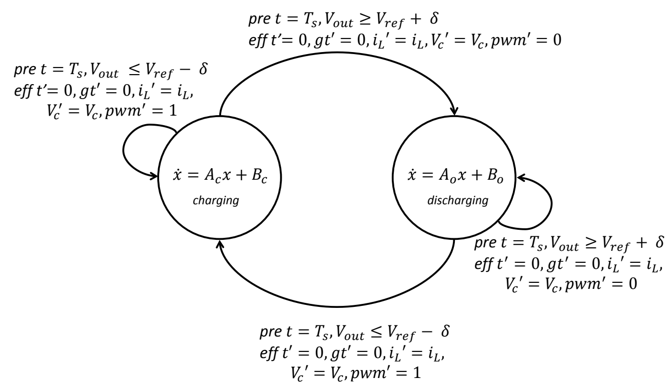

Fig. 4. Hybrid automaton model of the buck converter plant with a hysteresis controller.

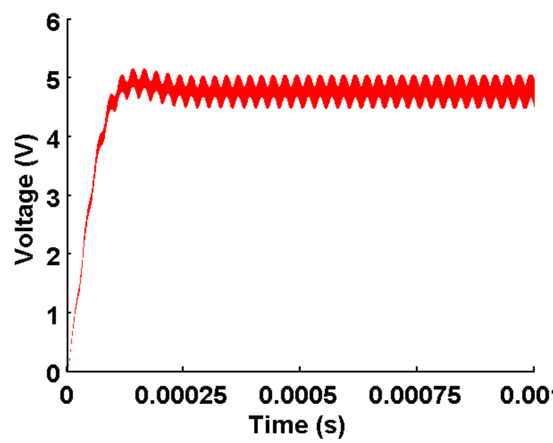

Fig. 5. Hysteresis controller: output capacitor voltage, V_c (V) vs. time, t(s), from SpaceEx.

with the hysteresis controller because it was effective in testing the closed-loop analysis capability of SpaceEx. Since the test was successful, we analyzed a more realistic pole-placement controller that was converted to an equivalent linear system controller as shown in Figure 2.

We first modeled the closed-loop system with a linear controller in Simulink, and achieved expected results as shown in Figure 3. However, after an extensive trial of tuning parameters in SpaceEx, we found it difficult to analyze such a system automatically by performing reachability computations. The user can tune a variety of parameters to make the reachability analysis more or less precise at the expense of runtime. For example, the user can choose the number of directions used in the support function representation, the sampling time (reachability time-step), or flow-pipe overapproximation tolerance. We found that with the linear controller, the combinations of both fast and slow dynamics, as well as the use of a relatively fast PWM period, made choosing such parameters difficult, even when analyzing the system from steady-state initial conditions (e.g., with the output voltage equal to the desired output voltage).

The PWM period of the system was 10^{-6}s, and we used a sampling period of 10^{-8}s in SpaceEx. The largest and smallest eigenvalues of A_{comp} differed by about four orders of magnitude (10^4). Particularly, the controller dynamics were much faster than the plant dynamics, and would cause the controller states to stabilize quicker than the plant states. This difference in magnitudes, however, made the choice of SpaceEx's sampling period quite small to avoid the overapproximation error from growing too large. Even with a choice of sampling time at 10^{-8}, the overapproximation error was so large that the system was in both the charging and discharging modes simultaneously, so the analysis was effectively useless. With this choice of sampling time, SpaceEx ran for about 20 minutes.

While perhaps an even smaller choice of sampling period or

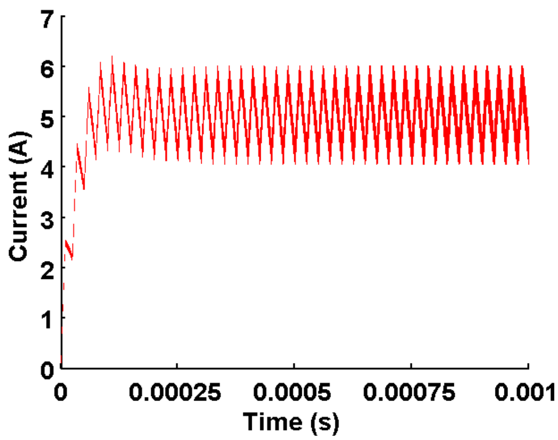

Fig. 6. Hysteresis controller: output inductor current, i_L (A) vs. time, t(s), from SpaceEx.

a larger number of directions would make the overapproximation error smaller, the increased runtime makes the analysis effectively infeasible. One potential solution would be to develop a method that can use variable time steps for different dimensions, particularly smaller time steps for dimensions with faster dynamics and larger time steps for dimensions with slower dynamics. Perhaps methods for handling dynamics of different speeds, such as time-scale separation, can be incorporated into reachability analysis to avoid the runtime and/or overapproximation error growth in such closed-loop systems [15].

V. Conclusion and Future Work

In this paper, we studied the use of reachability analysis for hybrid systems to verify properties of closed-loop power converters. We used the hybrid systems reachability tool SpaceEx to verify time-bounded voltage regulation of open-loop buck converters [7] as well as a closed-loop hysteresis controller model. Additionally, we analyzed limitations of verifying properties with more realistic controllers such as the equivalent linear controller for a pole-placement control design. The reachability analysis performed on the systems provides valuable information on the behavior of the converters. SpaceEx computes an overapproximation of the set of reachable states of the system and ensures that the system satisfies all desired safety properties for all possible executions. Therefore, both the open-loop system and hysteresis controller system satisfy the desired regulation property and can be deemed as robust designs. However, for the linear controller, our analysis exposed potential limitations in using reachability analysis. In particular, the combination of fast and slow dynamics appears to be challenging for current reachability methods that use a uniform time-step for all variables. For future work, this motivates new reachability methods that use non-uniform time-steps for different dimensions, which could possibly be detected automatically using the magnitude of the corresponding eigenvalue for a particular variable.

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