Stability of Digitally Interconnected Linear Systems

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Abstract— **A sufficient condition for stability of linear subsystems interconnected by digitized signals is presented. There is a digitizer for each linear subsystem that periodically samples an input signal and produces an output that is quantized and saturated. The output of the digitizer is then fed as an input (in the usual sense) to the linear subsystem. Due to digitization, each subsystem behaves as a switched affine system, where state-dependent switches are induced by the digitizer. For each quantization region, a storage function is computed for each subsystem by solving appropriate linear matrix inequalities (LMIs), and the sum of these storage functions is a Lyapunov function for the interconnected system. Finally, using a condition on the sampling period, we specify a subset of the unsaturated state space from which all executions of the interconnected system reach a neighborhood of the quantization region containing the origin. The sampling period proves to be pivotal—if it is too small, then a dwell-time argument cannot be used to establish convergence, while if it is too large, an unstable subsystem may not receive timely-enough inputs to avoid diverging.**

I. INTRODUCTION

Any controller implemented using a computer is subject to *digitization*—quantization, saturation, and sampling. Quantization and saturation arise from finite capacity and precision of digital communication and computation. Sampling arises due to finite capacity as well, but also from the fact that computation in sensing, communication, and actuation devices is driven by clock pulses. In this paper, we study linear systems interconnected by *digitizers*. A digitizer periodically samples its input signal and produces a quantized, saturated, and piecewise constant output signal. Thus, its output values come from a finite set after a known sampling delay. We believe that this notion of a digitizer captures a wide variety of sensors, actuators, computers, and communication channels. For interconnecting subsystems, we roughly follow the distributed control framework from [1], [2], where subsystems are interconnected over an arbitrary graph (see Fig. 1 for an example). With quantization, each linear subsystem can be viewed as an affine system, where the affine term exhibits state-dependent switching based on the quantized value. Further, the sampling causes new quantized values to arrive late.

The digitally interconnected system described above is analyzed in this paper by applying linear matrix inequality (LMI) techniques. First, storage functions are computed for subsystems as in [1], [3]. Then, for each quantization region in the interconnected state space, a Lyapunov function is computed which decreases so long as the digitizer continues to output the quantized value corresponding to that quantization region. Finally, a dwell-time argument using the sampling period of the digitizers establishes a notion of attractivity. Each linear subsystem and its digitizer are formalized as one hybrid input/output automaton (HIOA) [4], and the interconnected system is a composition of these automata, where the outputs of some automata are fed as inputs to other automata.

There is a large body of literature regarding quantization, saturation, and delay in control systems; we mention a few works that are closely related to our own. To the best of our knowledge, no works have addressed general interconnections with digitization that we consider here. A thorough overview of switched systems is available in the book [5], which also covers quantization and saturation, albeit under a different model where a single system undergoes quantization of input, output, and/or state feedback [6]. In [7], the authors apply LMI techniques to state and input feedbacks which are delayed, saturated, and quantized. In [8], the author presents methods for guaranteeing stability of piecewise affine systems, which can be viewed as switched affine systems that quantization naturally induces. The authors of [3] apply techniques from [8] to the interconnected framework from [1], [2] to show stability of piecewise affine interconnected systems. We apply a similar S-procedure used in [8], [3] and also [9] to restrict our search for Lyapunov functions to the domain of each quantization region (of which there are a finite number due to saturation). Unlike [8], [3], [9], we do not search for a common Lyapunov function which is continuous along switching surfaces, and instead find a Lyapunov function for each quantization region and then apply a dwell-time argument. The finite number of quantization regions in our model is similar to the use of a finite alphabet in the model of [10]. In [11], the author considers interconnections of hybrid systems and establishes input/output stability and small-gain results.

II. INTERCONNECTION AND DIGITIZATION MODELS

We consider N interconnected linear subsystems, where the output signals of some are fed to the input signals of others in the same way as in [3], [1], [2] (see, for example, Fig. 1). Unlike the prior work, however, here the input/output signals are *digitized*, that is, sampled, quantized, and saturated. To capture digitization, we group each linear subsystem with its input digitizer and model the combination

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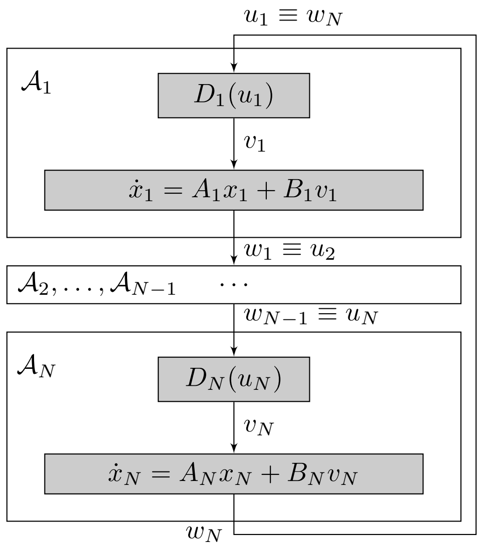

Fig. 1. Ring interconnection of N linear subsystems with digitization.

as a *Hybrid Input/Output Automaton (HIOA)* [4], [12] $\mathcal{A}_i$ for $i \in \{1, \ldots, N\}$.

A Linear Subsystem with Digitized Inputs: First, we describe the HIOA $\mathcal{A}_i$ of a single linear subsystem with digitized inputs; for reference, consider $\mathcal{A}_1$ in Fig. 1. $\mathcal{A}_i$ takes an input signal $u_i \in \mathbb{R}^m$, which is fed into the *digitizer* D_i. The digitizer output $v_i \in \Sigma^m$ where Σ is a finite subset of the reals, is called the *intermediate signal*, is quantized and saturated, and is only updated periodically. The intermediate signal v_i is the input (in the usual sense) to the linear differential equation $\dot{x}_i = A_i x_i + B_i v_i$. Finally, the output signal w_i from the linear subsystem is the output of automaton $\mathcal{A}_i$[1].

Quantization and saturation in the digitizer are modeled with a *quantization function*[2] $Q : \mathbb{R}^m \to \Sigma^m$. Sampling delays are captured with a timer as discussed below. Q induces a partitioning of $\mathbb{R}^m$ and the corresponding equivalence relation on $\mathbb{R}^m \times \mathbb{R}^m$ is denoted by $\sim$. The equivalence class of an element $x \in \mathbb{R}^m$ is denoted by $[x]$. For some set S, the quotient space is denoted $S_{\backslash\sim}$. The *preimage* of $v \in \Sigma^m$ is $Q^{-1}(v) \triangleq \{u \in \mathbb{R}^m : [u] = v\}$. Beyond a certain value $\pm M \in \Sigma$ called the *saturation range*, Q saturates and returns the same value. That is, for any $u \in \mathbb{R}^m$, if $\|u\|_\infty \geq M$, then $\|Q(u)\|_\infty = M$. We assume that when unsaturated, the difference between the quantizer's output and its input is bounded by a constant $\Delta > 0$. Formally, $\forall u \in \mathbb{R}^m$, if $\|Q(u)\|_\infty < M$, then $\|Q(u) - u\|_\infty \leq \Delta$. Due to the quantization error Δ, we cannot in general expect to have asymptotic stability in the usual sense where a system converges to an equilibrium as time goes to infinity. Similarly, because of the quantization saturation M, we will not be able to achieve global attractivity and instead will define an appropriate local region of attraction later.

For the remainder of the paper, we fix a sampling period $\phi > 0$. Automaton $\mathcal{A}_i$ with sampling period ϕ is a tuple $\langle \mathcal{V}_i, \mathcal{D}_i, \mathcal{T}_i \rangle$, where:

(i) $\mathcal{V}_i$: is the set of variables $\{x_i, w_i, u_i, v_i, h_i\}$, where: (a) *state variable* x_i takes values in $\mathbb{R}^n$, (b) *output variable* w_i takes values in $\mathbb{R}^m$, (c) *input variable* u_i takes values in $\mathbb{R}^m$, (d) *intermediate variable* v_i takes values in Σ^m, and (e) *timer variable* h_i takes values in $\mathbb{R}_{\geq 0}$. The *state space* $\mathcal{Q}_i$ is the set of all valuations of x_i, v_i, and h_i, that is, $\mathcal{Q}_i \triangleq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$. A *state* is denoted by bold $\mathbf{x}$. The set of valuations of x_i is denoted by the set $\mathcal{X}_i \triangleq \mathbb{R}^n$.

(ii) $\mathcal{D}_i \subseteq \mathcal{Q}_i \times \mathcal{Q}_i$ is a set of *transitions*. A transition $(\mathbf{x}, \mathbf{x}') \in \mathcal{D}$ is written as $\mathbf{x} \to_{\mathcal{A}_i} \mathbf{x}'$ or as $\mathbf{x} \to \mathbf{x}'$ when $\mathcal{A}_i$ is clear from context. There is a discrete transition $\mathbf{x} \to_{\mathcal{A}_i} \mathbf{x}'$ if and only if: (a) (Precondition) At the pre-state $\mathbf{x}$, ϕ time has elapsed since the previous discrete transition (i.e., $\mathbf{x}.h_i \geq \phi$) and the intermediate variable does not match the quantized input (i.e., $\mathbf{x}.v_i \neq Q(\mathbf{x}.u_i)$), and (b) (Post-state) All the variables' values in $\mathbf{x}'$ remain the same as in $\mathbf{x}$, except that the timer is reset (i.e., $\mathbf{x}'.h_i = 0$) and the intermediate variable is set to the quantized input (i.e., $\mathbf{x}'.v_i = Q(\mathbf{x}'.u_i)$). Thus, the discrete transitions model the switches in v_i caused by digitization.

(iii) $\mathcal{T}_i$: is the set of *trajectories* for the variables in $\mathcal{V}_i$, which models the continuous evolution of the variables over time intervals. Specifically, for $T \geq 0$, a T-trajectory is a function $\tau : [0, T] \to \mathcal{Q}_i$ such that $\forall t \in [0, T]$ we have: (a) $\tau(t).h_i = \tau(0).h_i + t$: the timer grows monotonically at unit rate, (b) $\tau(t).v_i = \tau(0).v_i = Q(\tau(0).u_i)$: the intermediate variable remains constant, and (c) for any trajectory ζ of the intermediate variable v_i, $\tau(t).x_i$ is obtained by integrating the linear differential equation $A_i \tau(t).x_i + B_i \zeta(t)$.[3]

The domain for a trajectory $\tau \in \mathcal{T}_i$ is denoted by $\tau.\mathsf{dom}$. We define $\tau.\,\mathsf{ltime}$ as the right endpoint of $\tau.\mathsf{dom}$, $\tau.\,\mathsf{lstate} \triangleq \tau(\tau.\,\mathsf{ltime})$, and $\tau.\,\mathsf{fstate} \triangleq \tau(0)$. The discrete-continuous behavior of a HIOA is defined in terms of executions. An *execution* of $\mathcal{A}_i$ is a finite or infinite sequence of trajectories $\tau_0\ \tau_1\ \ldots$, such that for all indices k in the sequence, there is a discrete transition $\tau_k.\,\mathsf{lstate} \to \tau_{k+1}.\,\mathsf{fstate}$.

Interconnected System as Composition of HIOAs: The interconnected system is another HIOA called System which is formally defined as the composition of several $\mathcal{A}_i$'s [4], [12]. The interconnection is specified by a function G which maps the output variables of each automaton to the input variable of some automaton in the system. For a regular interconnection, for example a ring (see Fig. 1), $G(u_i) \triangleq$

[1]We suppose $w_i = x_i$ for clarity of presentation in this paper, but nothing prevents the more general case $w_i = C_i x_i$.

[2]For simplicity, we assume that each $\mathcal{A}_i$ uses the same digitizer D_i, each of which uses the same quantization function Q.

[3]The solution $\tau(t).x_i$ is well-defined even if the input trajectory $\zeta(t)$ is unknown, so long as $\zeta(t)$ is integrable, which is the case because ζ is piecewise constant. $\mathcal{T}_i$ also satisfies the stopping condition: $\forall t \in [0, T]$, if $\tau(t).h_i \geq \phi$ and $Q(\tau(t).u_i) \neq \tau(t).v_i$, then $t = T$, that is, t must be the endpoint of the trajectory. This condition, forces the intermediate variable v_i to change once the actual input u_i does not match the quantized input and $\tau.h_i \geq \phi$.

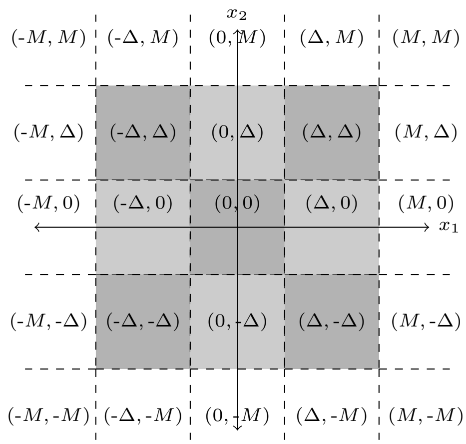

Fig. 2. Equivalence classes of the quantized state space $\mathcal{X}_{\backslash\sim}$ are squares projected onto the real plane state space $\mathcal{X}$ for two interconnected one-dimensional systems. Example quantizer output for each equivalence class is indicated, as well equivalence classes beyond the quantization saturation M. There are 9 unsaturated modes in $\mathcal{M}_{\backslash\sim}$.

$w_{(i-1 \mod N)+1}$. In general, G specifies the interconnection as some arbitrary graph [1], and we assume all dimensions are compatible.

For the composed HIOA System, the set of variables $\mathcal{V}$ is the union of each $\mathcal{V}_i$ for $i \in \{1, \ldots, N\}$. We write x, v, etc., for the stacked vectors of x_i, v_i, etc. The state space $\mathcal{Q}$ is the product of each $\mathcal{Q}_i$. $\mathcal{X}$ is defined to be the product of each $\mathcal{X}_i$. We abuse notation and write states of System as $\mathbf{x}$. A discrete transition occurs in System iff at least one of the automata $\mathcal{A}_i$ in the composition has a discrete transition. Along each trajectory of System, all the non-input variables of $\mathcal{A}_i$ flow according to the conditions defined for $\mathcal{T}_i$, and the input variables flow according to the trajectories defined for $G(u_i)$ in the corresponding automaton. Each of these trajectories must stop when the output of some automaton $\mathcal{A}_i$ crosses a quantization partition *and* the corresponding timer $h_i \geq \phi$.

Based on the interconnection G, all the quantizers together induce an equivalence relation $\sim$ on the interconnected state space $\mathcal{X}$. Let $\mathcal{X}_{\backslash\sim}$ be the set of all such equivalence classes on $\mathcal{X}$ under $\sim$. By abuse of notation, we will write the preimage as $Q^{-1}(q)$ for $q \in \mathcal{X}_{\backslash\sim}$ and $[x]$ as the equivalence class of $x \in \mathcal{X}$. Let $\mathcal{M} \triangleq \{x \in \mathcal{X} : \|x\|_\infty \leq M\}$ be the unsaturated state space, which contains all unsaturated points in the interconnected state space, and $\mathcal{M}_{\backslash\sim}$ the corresponding quotient under $\sim$. See Fig. 2 for examples of $\mathcal{X}$, $\mathcal{X}_{\backslash\sim}$, $\mathcal{M}$, and $\mathcal{M}_{\backslash\sim}$. Note that the system obtained by interconnecting the linear subsystems directly as in [1], without using the digitizers has an equilibrium point at $\mathbf{0} \in \mathcal{X}$. With digitization, however, there may be multiple equilibria (see Fig. 4). We assume that the intermediate signal v_i is $\mathbf{0} \in \mathcal{X}_i$ when the input signal u_i is in $Q^{-1}([\mathbf{0}])$, the preimage of the equivalence class containing the origin.

III. Stability Analysis

In this section, we establish a notion of stability for the digitally interconnected system. It is impossible to ensure the usual asymptotic stability where $x \to 0$ as $t \to \infty$ due in part to the quantization error Δ and also because within $Q^{-1}([\mathbf{0}])$, the input to any subsystem i with unstable system matrix A_i is zero and therefore i will be unstable. Instead, we construct a Lyapunov-like function for a subset of the unsaturated quantization regions. Using these functions and a dwell-time argument, we show (Theorem 1) that any infinite execution of System starting in the *terminable set* Λ (defined below) reaches and remains within the *final set* Ω (defined below) which contains the preimage of the equivalence class of the equilibrium point(s) of System.

We begin by constructing, for each unsaturated quantization region, a subset of the state space over which we can construct a Lyapunov function. Intuitively, if we considered quantization and saturation without delays, the switching surfaces would be the boundaries of the quantization regions. However, in our model, the switching surfaces are not necessarily the boundaries of the quantization regions: due to the sampling delay ϕ enforced by the timer h_i, there is a continuum of switching surfaces. That is, based on the starting state of a trajectory, the switch occurs whenever the conditions for the timer to reset are satisfied, which could be at the boundary of two quantization regions, or potentially elsewhere in the new quantization region (see Fig. 3 for an example trajectory illustrating this).

We say that *the input is fixed to* $q \in \mathcal{X}_{\backslash\sim}$ if for a state $\mathbf{x}$, we have $\mathbf{x}.x \notin Q^{-1}(q)$, $\mathbf{x}.v = q$, and $\forall i \in \{1, \ldots, N\}$, $\mathbf{x}.h_i < \phi$. This captures the notion that subsystems are using the quantized value from an equivalence class that the system state is no longer in due to sampling delay. For each $q \in \mathcal{M}_{\backslash\sim}$, let $B_{q,\phi} \subseteq \mathcal{X}$ be a set of states containing all states of System that can be reached from any point in $Q^{-1}(q)$ within ϕ time by following the trajectories with the input fixed to q. Since the unsaturated quantization error is bounded (by Δ), every unsaturated quantization region $Q^{-1}(q)$ is a bounded set, and so is $B_{q,\phi}$. For each $q \in \mathcal{M}_{\backslash\sim}$, we can compute an ellipsoid $\mathcal{E}_q \triangleq \{x \in \mathcal{X} : (x-g)^T R_q (x-g) \leq 1\}$ containing $Q^{-1}(q)$ and $B_{q,\phi}$, where g is the centroid of $Q^{-1}(q)$ and $R_q \in \mathcal{X} \times \mathcal{X}$ is a symmetric, positive definite matrix (see, e.g. [13, Section 5.2] for an algorithm). So we have $\mathcal{E}_q \supseteq B_{q,\phi} \supseteq Q^{-1}(q)$, that is, $\mathcal{E}_q$ contains the preimage $Q^{-1}(q)$ of the equivalence class q, and all states that may be reached following a trajectory of up to ϕ time from any point $x \in Q^{-1}(q)$ with the input q fixed.

Now we state a lemma giving a condition on when an ellipsoid contains another ellipsoid, which is used in the S-procedure in the LMI formulated below.

Lemma 1: [9, Lemma 11.6] For $a \in \{1, 2\}$, $g_a \in \mathcal{X}$, and a symmetric, positive definite matrix $P_a \in \mathcal{X} \times \mathcal{X}$, let $\mathcal{E}_a \triangleq \{x \in \mathcal{X} : (x - g_a)^T P_a (x - g_a) \leq 1\}$ be an ellipsoid centered at g_a. If $\exists \eta \geq 0$ such that $0 \succeq$

$$\begin{bmatrix} P_2 & -P_2 g_2 \\ -g_2^T P_2 & g_2^T P_2 g_2 - 1 \end{bmatrix} - \eta \begin{bmatrix} P_1 & -P_1 g_1 \\ -g_1^T P_1 & g_1^T P_1 g_1 - 1 \end{bmatrix}$$

then, $\mathcal{E}_1 \subset \mathcal{E}_2$.

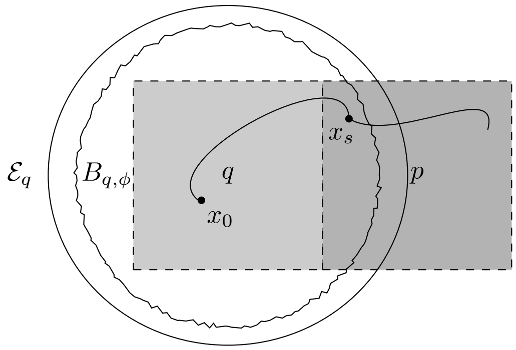

Fig. 3. Example trajectory for an interconnected system with a two-dimensional state space. The trajectory starts from x_0, but the sampling delay ϕ causes the input v to remain fixed to q even though the trajectory has entered the quantization region p. The update to $v = p$ occurs at x_s instead of at the boundary between p and q. The sets $\mathcal{E}_q$ and $B_{q,\phi}$ over which the Lyapunov function V_q is valid are shown.

Construction of Lyapunov Functions: We now construct a Lyapunov function for a subset of the unsaturated quantization regions. We begin by constructing storage functions for each linear subsystem, which we will then sum to yield the Lyapunov function. For each $\mathcal{A}_i$ with input from $\mathcal{A}_j$, we consider a quadratic storage function $V_i(x_i) = x_i^T P_i x_i$ where $V_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$, for some symmetric, positive definite matrix $P_i \in \mathcal{X}_i \times \mathcal{X}_i$. Its derivative along the trajectories is given by $\dot{V}_i(x_i) =$,

$$\dot{x}_i^T P_i x_i + x_i^T P_i \dot{x}_i = \begin{bmatrix} x_i \\ Q(x_j) \end{bmatrix}^T Y_i \begin{bmatrix} x_i \\ Q(x_j) \end{bmatrix}, \quad (1)$$

$$\text{where } Y_i \triangleq \begin{bmatrix} A_i^T P_i + P_i A_i & P_i B_i \\ B_i^T P_i & 0 \end{bmatrix}.$$

For V_i to be a Lyapunov function we would require $\dot{V}_i(x_i) < 0$, but as it is a storage function we require the weaker condition $\dot{V}_i(x_i) < s_i(u_i, w_i)$, where s_i is called the *supply rate*. A sufficient condition for the sum of V_i and V_j to yield a Lyapunov function for the interconnected system is that $s_i(u_i, w_i) = -s_j(u_j, w_j)$.

Next we assume that supply rates s between interconnected subsystems $\mathcal{A}_i$ and $\mathcal{A}_j$ are quadratic. That is, the supply rate for all $i, j \in \{1, \ldots, N\}$ such that $\mathcal{A}_i$'s input is connected to $\mathcal{A}_j$'s output by the interconnection G, we have

$$s_i(u_i, w_i) = \begin{bmatrix} w_i \\ u_i \end{bmatrix}^T Z_i \begin{bmatrix} w_i \\ u_i \end{bmatrix}, \text{ and}$$

$$Z_i = -\begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} Z_j \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix},$$

for symmetric Z_i, and the n-dimensional identity matrix I_n. This enforces that supply rates satisfy $s_i = -s_j$. The earlier Lemma 1 ensures that the sum of the storage functions V_i is a Lyapunov function for System over each $\mathcal{E}_q$.

In order to find storage functions for each subsystem, we formulate the following LMI for each $\mathcal{A}_i$,

$$P_i = P_i^T > 0 \text{ and } 0 \succ Y_i, \quad (2)$$

where Y_i is from (1). Additionally, the LMI has the following constraints corresponding to each interconnected $\mathcal{A}_i$ and $\mathcal{A}_j$ to enforce $s_i = -s_j$ over the ellipsoid $\mathcal{E}_q$. For each $q \in \mathcal{M}_{\setminus\sim}$,

$$\eta \geq 0, \quad (3)$$
$$0 \succ Y_i - Z_i - \eta \begin{bmatrix} R_q & -R_q q \\ -q^T R_q & q^T R_q q - 1 \end{bmatrix}, \text{ and}$$
$$0 \succ Y_j - Z_j - \eta \begin{bmatrix} R_q & -R_q q \\ -q^T R_q & q^T R_q q - 1 \end{bmatrix}.$$

If this LMI is feasible[4], for each $q \in \mathcal{M}_{\setminus\sim}$, we get a Lyapunov-like candidate $V_q : \mathcal{X} \to \mathbb{R}_{\geq 0}$, where

$$V_q(x) = \sum_{i=1}^{N} x_i^T P_i x_i,$$

for $x \in \mathcal{X}$ and $x_i \in \mathcal{X}_i$. The candidate V_q decreases along trajectories of System, so long as the state x_i remains within the ellipsoid $\mathcal{E}_q$ which contains $Q^{-1}(q)$, and the input remains fixed to q.

We next define the terminable set as

$$\Lambda \triangleq Q^{-1}(\{q \in \mathcal{X}_{\setminus\sim} : \exists c \in \mathbb{R}_{\geq 0} \text{ such that } Q^{-1}(q) \subseteq \mathcal{L}_{q,c}(x) \subseteq \mathcal{M}\}),$$

where $\mathcal{L}_{q,c}(x) \triangleq \{x \in \mathcal{X} : V_q(x) \leq c\}$ is the c-sublevel set of $V_q(x)$. In other words, Λ is the set of quantization regions for which there is some sublevel set which (a) contains the region itself, and (b) is entirely contained in the unsaturated state space $\mathcal{M}$. Indeed, some unsaturated quantization regions may not be in Λ (see Fig. 4).

The next lemma states that V_q is a Lyapunov function and the sublevel sets of V_q are invariant for System so long as the input is fixed to q. If a sublevel set has points outside the unsaturated region, then we cannot guarantee that a trajectory will not leave the unsaturated region (see Fig. 4).

Lemma 2: For any equivalence class $q \in \mathcal{M}_{\setminus\sim}$ except $[\mathbf{0}]$ such that $Q^{-1}(q) \subseteq \Lambda$, if $\mathbf{x}.x \in B_{q,\phi}$ and $\mathbf{x}.v = q$ (the input is fixed to q), then $\dot{V}_q(\mathbf{x}.x) < 0$.

Bounding the Increase of V_q from Switching: When switching inputs between equivalence classes $q, p \in \mathcal{M}_{\setminus\sim}$, the corresponding Lyapunov functions may not be equal. In particular, the value may be larger, $V_p(\mathbf{x}.x) > V_q(\mathbf{x}.x)$ at a state $\mathbf{x}$ where the digitized input changes from q to p. We bound the increase by a factor μ as follows

[4] A condition to guarantee feasibility is that the interconnected system without digitization, i.e., as in the framework from [1], is exponentially stable.

(see Fig. 3). Define the maximum switching factor between any unsaturated quantization regions $q, p \in \mathcal{M}_{\backslash\sim}$ as

$$\mu \triangleq \max_{q,p\in\mathcal{M}_{\backslash\sim}} \max_{B_{q,\phi}\cap Q^{-1}(p)} \frac{V_p(\mathbf{x}.x)}{V_q(\mathbf{x}.x)}. \quad (4)$$

By Lemma 2 we have a Lyapunov function V_q for any quantization region $q \in \mathcal{M}_{\backslash\sim}$ except $[\mathbf{0}]$ satisfying $Q^{-1}(q) \subseteq \Lambda$, so we determine a minimum convergence rate of any V_q as follows. Consider any trajectory τ such that $\exists\lambda_q > 0, \forall\tau.x \in B_{q,\phi}, V_q(\tau(t).x) \leq V_q(\tau(0).x)e^{-\lambda_q t}$. Then the minimum convergence rate is

$$\lambda_m \triangleq \min_{q\in(\mathcal{M}_{\backslash\sim}-[\mathbf{0}])\cap\Lambda} \lambda_q. \quad (5)$$

We recall from [14, Lemma 2] that under arbitrary switching, if each mode of a switched linear system is exponentially stable, then one can pick the dwell-time ϕ sufficiently large so that the switched system is exponentially stable for any switching signal which dwells in each mode for at least ϕ time. Particularly, we assume $\phi > \frac{\log\mu}{2\lambda_m}$, where this is the weaker average dwell-time constant.

We define the *final set of states* as

$$\Omega \triangleq \cup_{q\in\mathcal{M}_{\backslash\sim}} \mathcal{L}_{q,c}(\mathbf{x}.x),$$

where for each q, c is chosen such that $\mathcal{L}_{q,c}(\mathbf{x}.x)$ is the *smallest* (in terms of containment) sublevel set of the corresponding V_q containing the set $B_{[\mathbf{0}],\phi}$, so $\mathcal{L}_{q,c}(\mathbf{x}.x) \supseteq B_{[\mathbf{0},\phi]} \supseteq Q^{-1}([\mathbf{0}])$. For any of these q, note that it is not necessary that $Q^{-1}(q) \subseteq \mathcal{L}_{q,c}$, only that $Q^{-1}([\mathbf{0}]) \subseteq \mathcal{L}_{q,c}$, that is, the level sets are not excessively large. The next lemma says any execution starting from Ω cannot leave Ω.

Lemma 3: If $\Omega \subseteq \Lambda$, then Ω is invariant.

Proof: Consider any execution starting with a state $\mathbf{x}.x \in \Omega$. By the assumption that $\Omega \subseteq \Lambda$, we have that for any $q \in \mathcal{M}_{\backslash\sim} \cap \Lambda$ except $[\mathbf{0}]$, V_q is a Lyapunov function and has invariant level sets by Lemma 2. So for the equivalence class p such that $\mathbf{x}.v = p$, the level set $\mathcal{L}_{p,c}(\mathbf{x}.x)$ is invariant, and hence Ω is invariant since p is included in the union. ■

The following theorem states a local attractivity property, that from any point in the terminable set Λ, eventually a state is visited in the final set of states Ω.

Theorem 1: Suppose the sampling period $\phi > \frac{\log\mu}{2\lambda_m}$. If $\Omega \subseteq \Lambda$, then any infinite execution α of System starting in Λ eventually reaches Ω.

We remark that requiring $\Omega \subseteq \Lambda$ sets a lower-bound on the saturation range M and an upper-bound on the sampling period ϕ. If M is too small, then subsystems may not receive large enough stabilizing inputs, and if ϕ is too large, then subsystems may not receive stabilizing inputs fast enough.

Proof: By the assumption that $\Omega \subseteq \Lambda$ and $\alpha.\mathsf{fstate} \in \Lambda$, any infinite execution α starts with a state $\mathbf{x}$ in the preimage $Q^{-1}(q)$ of an equivalence class $q \in \mathcal{M}_{\backslash\sim} - \{[\mathbf{0}]\}$, where the corresponding Lyapunov function V_q satisfies Lemma 2. Since $\dot{V}_q(\mathbf{x}.x) < 0$ when $\mathbf{x}.x \in B_{q,\phi}$, there is a state $\mathbf{x}' \in \alpha$ appearing after $\mathbf{x}$ such that $\mathbf{x}'.x \notin Q^{-1}(q)$.

There are two cases. First, the preimages of a finite sequence of distinct equivalence classes $q_1, q_2, \ldots, q_a \in \mathcal{M}_{\backslash\sim} \cap \Lambda$ are visited by states following $\mathbf{x}'$ in α, where the Lyapunov functions for each of these decreases by Lemma 2. By following such a sequence, eventually a state $\mathbf{x}_t$ appears after $\mathbf{x}'$ such that $\mathbf{x}_t.x \in \mathcal{L}_{q_a,c}(\mathbf{x}_t.x) \subseteq \Omega$. Otherwise, the preimages of a sequence of equivalence classes $q_1, q_2, \ldots, q_a \in \mathcal{M}_{\backslash\sim}$ are visited containing a cycle, so suppose $q_1 = q_a$ and $Q^{-1}(q_a) \notin \Omega$. We note that the number of mode switches is at least a. We now eliminate this case from occurring indefinitely by contradiction. Let $\mathbf{x}'$ be a state such that $\mathbf{x}'.x \in Q^{-1}(q_1)$ and let $\mathbf{x}''$ be a state such that the equivalence class is visited in the cycle, so $\mathbf{x}''.x \in Q^{-1}(q_1)$. Using (4) and the number of mode switches, we have $V_{q_1}(\mathbf{x}''.x) \leq \mu^a V_{q_1}(\mathbf{x}'.x)$ and using (5), we have $V_{q_1}(\mathbf{x}''.x) \leq V_{q_1}(\mathbf{x}'.x)e^{-\lambda_m T}$, for $T \geq a\phi$, since due to sampling, any trajectory dwells between mode switches for at least ϕ time. Now, if $V_{q_1}(\mathbf{x}''.x) > V_{q_1}(\mathbf{x}'.x)$, then it must be the case that $T < a\phi$, a contradiction that $\phi > \frac{\log\mu}{2\lambda_m}$. Thus, any infinite execution must have a state appearing after $\mathbf{x}'$ in Ω. ■

IV. EXAMPLE

We now describe an example illustrating the methodology presented in the paper. For all resulting Lyapunov functions, we formulated LMIs in the Yalmip [15] interface to the solver SeDuMi [16] in Matlab. The example is a ring interconnection of two one-dimensional linear subsystems (see Fig. 1 and instantiate $N = 2$). One linear subsystem is stable, the other is unstable, and the interconnection without digitization is Hurwitz. The output w_1 from $\mathcal{A}_1$ is the input u_2 to $\mathcal{A}_2$, which then quantizes and saturates u_2 to the intermediate variable $v_2 = Q(u_2)$ at the sampling times ϕ and vice-versa. The linear system parameters are $a_1 = -2$, $b_1 = -3$, $a_2 = 1$, and $b_2 = 1$. Without digitization, the interconnected system can be modeled as one linear system $\dot{x} = Ax$, where $A = \begin{bmatrix} -2 & -3; & 1 & 1 \end{bmatrix}$, with eigenvalues $\lambda = -\frac{1}{2} \pm \frac{\sqrt{2}}{3}\iota$ for $\iota = \sqrt{-1}$. Thus it is globally exponentially stable with an equilibrium point at the origin, which satisfies all assumptions made in Sections II and III. Each subsystem uses the quantization function

$$Q(u_i) \triangleq \begin{cases} \Delta\,\mathrm{sgn}\,(u_i)\left\lfloor \frac{|u_i|}{\Delta} \right\rfloor, & \text{if } -M \leq u_i \leq M, \\ M, & \text{if } u_i > M, \text{ and} \\ -M, & \text{otherwise,} \end{cases}$$

where $\mathrm{sgn}\,(\cdot)$ is the sign function, $\lfloor\cdot\rfloor$ is the floor, $M \in \mathbb{R}$ is the saturation constant, and $\Delta \in \mathbb{R}_+$ is the maximum error. For the simulations, we fixed $\Delta = 1$ and $M = 3$. For these parameters, Q takes values from the set $\Sigma = \{-3, -2, -1, 0, 1, 2, 3\}$, and we have $Q(0) = 0$. For this Q, we have $Q^{-1}([\mathbf{0}]) = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq \frac{1}{2}\}$. While there are 49 total equivalence classes in $\mathcal{X}_{\backslash\sim}$, there are only 25 unsaturated equivalence classes in $\mathcal{M}_{\backslash\sim}$ (that is, excluding -3 and 3 from Σ), so we formulate 25 LMIs from (2) and 3), yielding 25 Lyapunov functions V_q.

In Fig. 4, the sampling period $\phi = 0.001$ and the terminable set Λ is visualized by the quantization regions

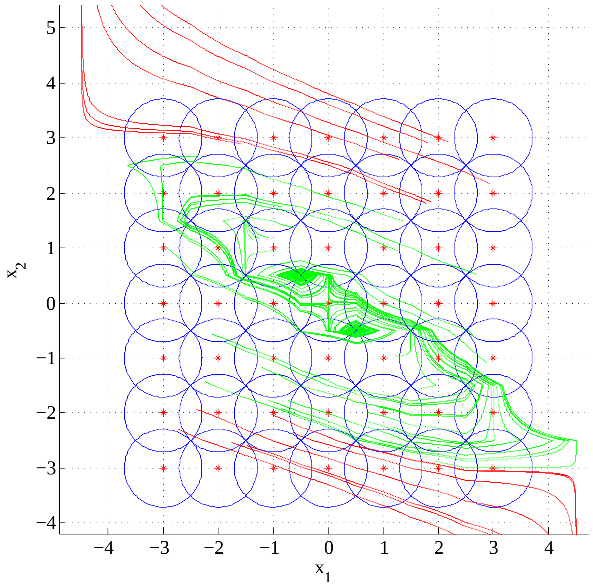

Fig. 4. Trajectories illustrating terminable set Λ and final set of states Ω. About 50 trajectories are shown, with those entering Ω in green, while those that diverge due to saturation are in red. Blue circles are ellipsoids containing the square equivalence classes defined by the quantizer. Red stars are quantizer values.

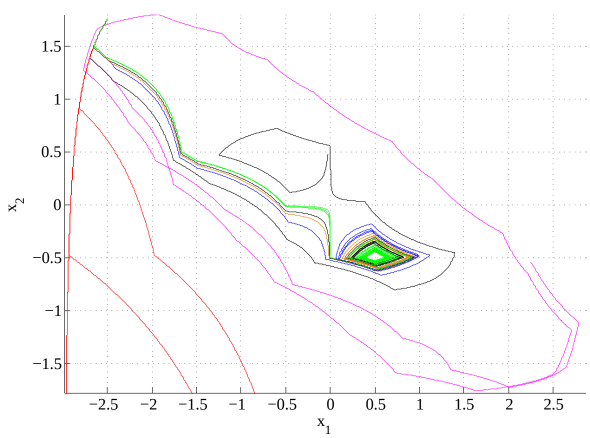

Fig. 5. Trajectories for increasing values of sampling period ϕ from the same initial condition. Trajectories which reach Ω are shown in colors other than red, while those that diverge due to the sampling period being too large are in red.

from which every trajectory converges. Observe that there are two equilibria with stable limit cycles, one at $(-\frac{\Delta}{2}, \frac{\Delta}{2})$ and another at $(\frac{\Delta}{2}, -\frac{\Delta}{2})$. In Fig. 5, the sampling period ϕ was increased from 0.001 according to $2^k 0.001$ where $k \in \{1, \ldots, 10\}$ is the simulation iteration. Trajectories all began from the same initial condition. Increasing ϕ causes the size of the subset of the state space contained by the limit cycle to increase, so Ω grows with k. This continues up until the sampling period ϕ is so large that the unstable subsystem does not receive timely enough stabilizing input and diverges, violating the assumption $\Omega \subseteq \Lambda$.

V. Conclusion

In this paper, we presented a dwell-time based sufficient condition on Lyapunov functions constructed for quantization regions to establish a form of stability of an interconnected system composed of linear subsystems connected through digitizers that have quantization, saturation, and sampling delay. We would like to study alternative techniques of establishing stability in interconnected systems, perhaps by using piecewise-quadratic common Lyapunov functions [17], [18], [3] while also accounting for sampling delays by perhaps treating the delay as a disturbance. It would also be interesting if some regularity of the quantized state space can be exploited to reduce the number of LMIs being solved.

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