Verification of Distributed Cyber-Physical Systems:
Stability of Digitally Interconnected Linear Systems

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Introduction

- Consider the problem of proving that a group of mobile robots in the plane, each following the same protocol, eventually forms an equi-spaced formation.
- Cyber-physical systems (CPS) have a strong coupling between computational and physical processes, like the robot system.
- This work focuses on algorithmic techniques for automatically proving stability for a class of CPS with linear dynamics.

Interconnected Systems

An interconnected system [1] is composed of N linear subsystems, where output signals of some systems are fed as input signals to other systems according to an interconnection function G.

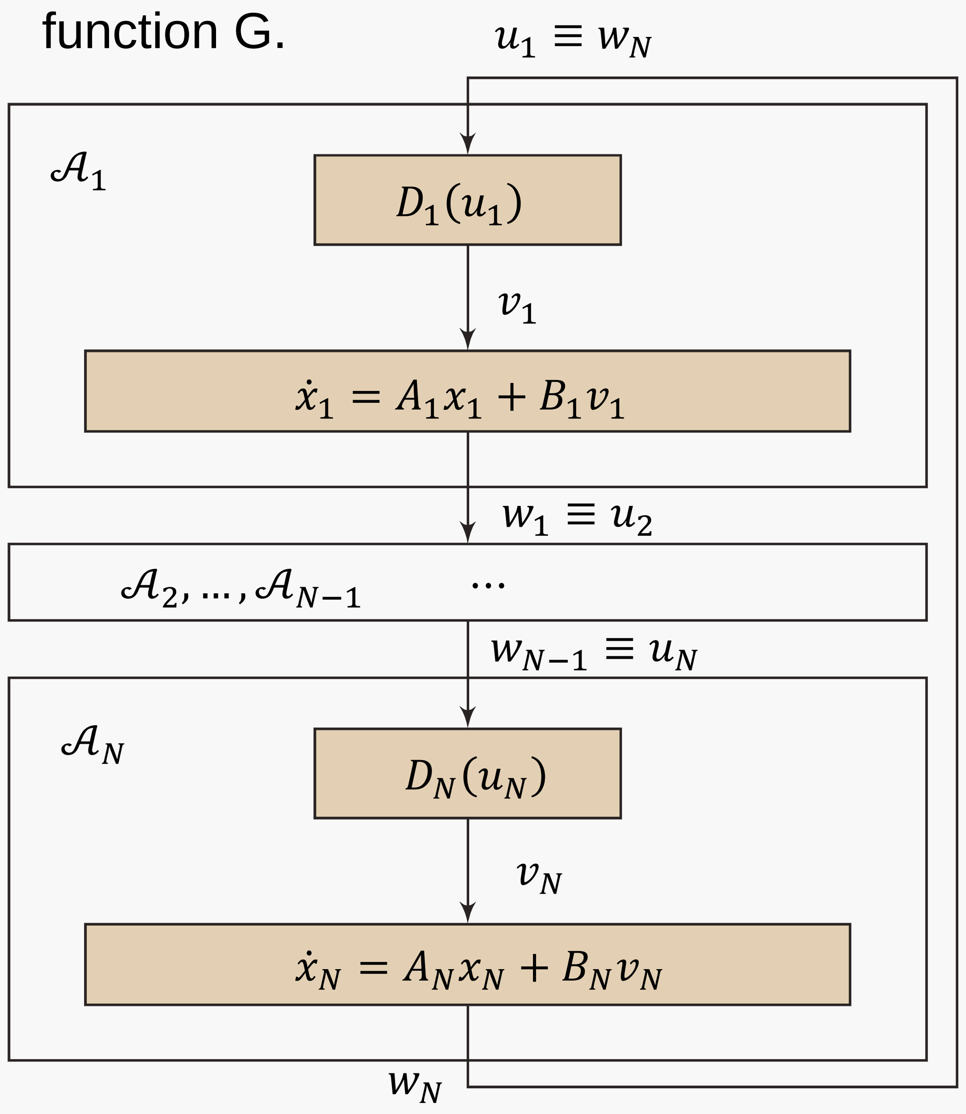


Fig. 1 (left): Ring interconnection of N linear subsystems with digitization defined by an interconnection function G. Each linear subsystem and its digitizer D_i is modeled as a hybrid input/output automaton denoted $\mathcal{A}_i$.

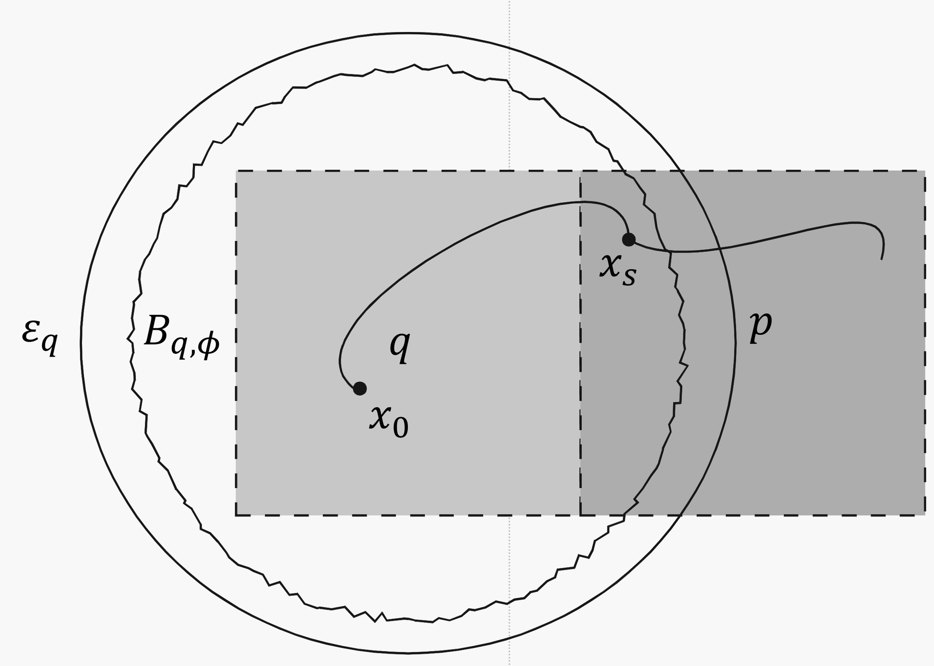


Fig. 3 (left): Illustration of digitization on an example execution for an interconnected system with a two-dimensional state space. The trajectory starts from x_0, but the sampling delay ϕ causes the input v to remain fixed to q even though the trajectory has entered the quantization region p. The update to v = p occurs at x_s instead of at the boundary between p and q. The sets E_q and $B_{q,\phi}$ over which the Lyapunov function V_q is valid are shown.

Digitization

Enforces three constraints on the inputs to subsystems.

1. **Quantization** and **Saturation**: the input takes values from a finite subset of the reals (Fig. 2).
   - If unsaturated, the quantized value is near the actual value.
2. **Sampling**: an input signal may only change values periodically (Fig. 3).

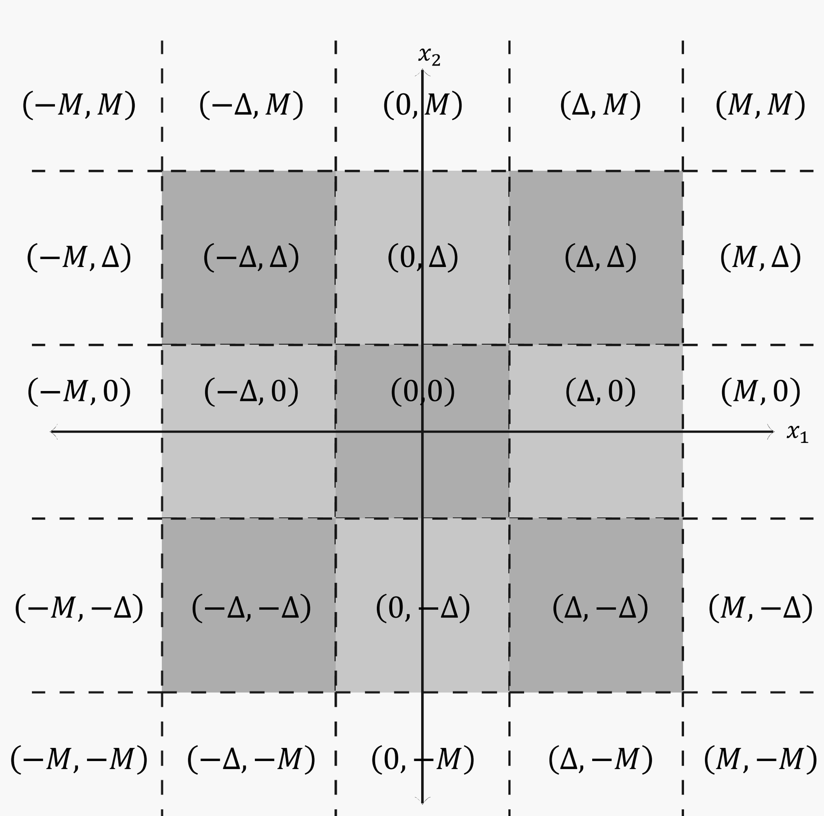


Fig. 2 (left): Example of two interconnected one-dimensional subsystems, where quantization regions (equivalence classes) are squares projected onto the real plane state space. Quantizer output for each equivalence class is indicated where Δ is a constant. There are 9 unsaturated quantization regions, and 16 saturated regions beyond the quantization saturation M.

Stability Analysis

- Compute **Lyapunov function V_q** for each unsaturated quantization region q by solving linear matrix inequalities (LMIs).
  - Compute **overapproximation $B_{q,\phi}$ of states reachable from quantization region q** (Figs. 3 and 4).
  - Storage function V_i for each subsystem.
  - Extra constraint if subsystems i and j are connected according to G.
  - Compute an ellipsoid E_q containing $B_{q,\phi}$.
  - Apply the **S-procedure** to force the domain of the Lyapunov function to be E_q.
  - If feasible, then $V_q = \Sigma_{i \in \{1,\ldots,N\}} V_i$ is a Lyapunov function.

Stability Results

- Due to saturation, define a **local region of attraction Λ**.
  $\Lambda \triangleq Q^{-1}\left(\{q \in \mathcal{X}_{\setminus\sim} : \exists c \in \mathbb{R}_{\geq 0} \text{ such that } Q^{-1}(q) \subseteq \mathcal{L}_{q,c}(x) \subseteq \mathcal{M}\}\right)$
  - Elements of the state space such that there is some sublevel set that contains the quantization region, and is entirely contained in the unsaturated state space.
- Due to quantization, **establish convergence to the set Ω**.
  $\Omega \triangleq \bigcup_{q \in \mathcal{M}_{\setminus\sim}} \mathcal{L}_{q,c}(x)$
  - The union, over each unsaturated quantization region, of the smallest sublevel sets of the corresponding Lyapunov function containing the quantization region that contains the origin.

Lemma: For any unsaturated quantization region q in Λ (except the origin), if $x \in B_{q,\phi}$ and v = q, then V_q is a Lyapunov function.

Theorem: If the sampling period $\phi > (\log \mu) / 2\lambda_m$ and $\Omega \subset \Lambda$, then any infinite execution starting in Λ eventually reaches and remains in Ω.

- Restriction on ϕ ensures that Lyapunov functions decrease enough between switches (average dwell-time constant).
- λ_m: minimum convergence rate over all quantization regions.
- μ: maximum switching factor increase between any two unsaturated quantization regions.

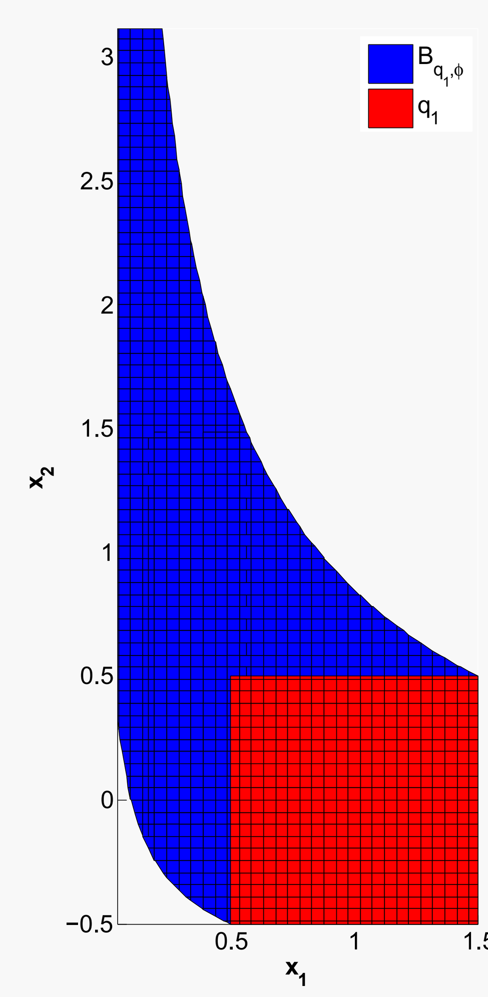
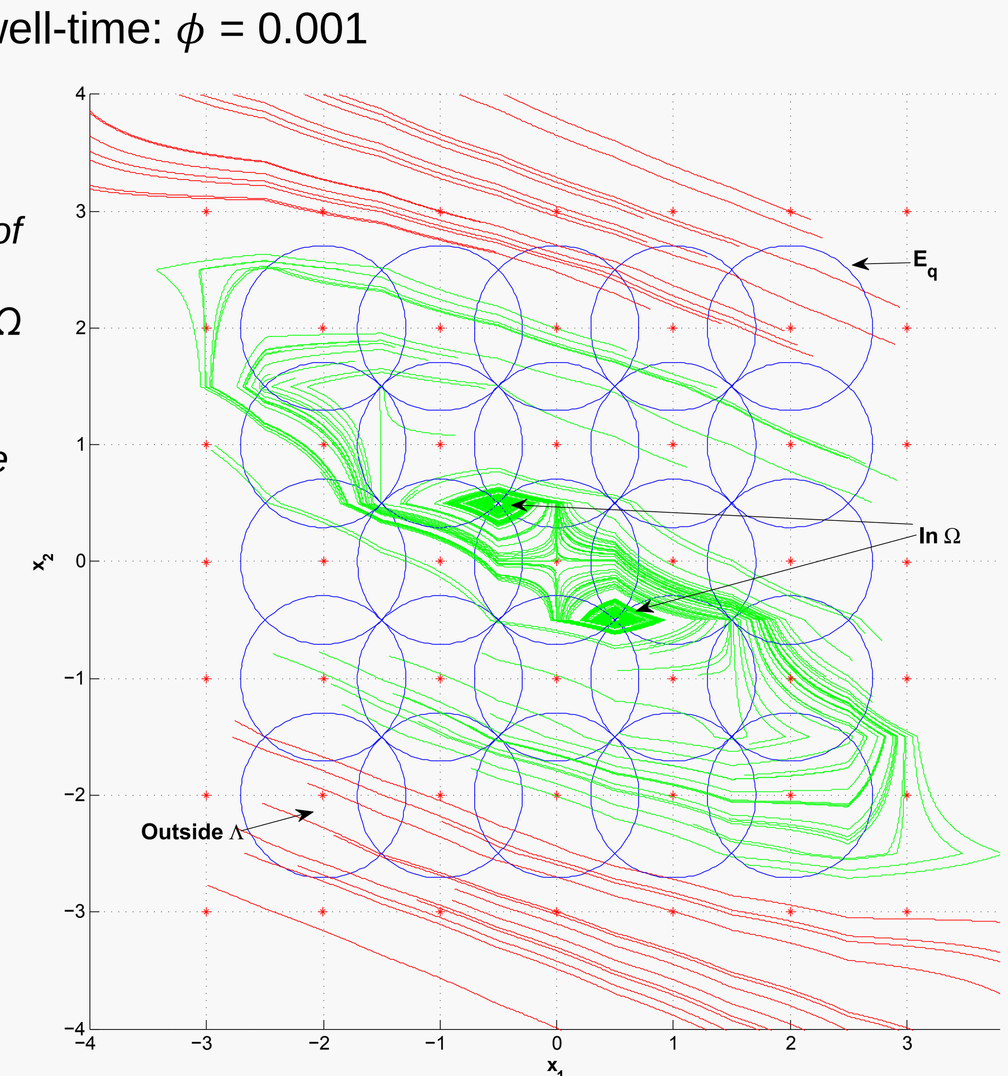


Fig. 4 (left): Computation of $B_{q,\phi}$ using the tool from [2] for a quantization region of the example. Sampling time is chosen to be much larger (ϕ =1 instead of 0.001) than it usually would be for illustration.

Example

- Two 1-dimensional linear systems connected in a ring (like Fig. 1).
- Parameters: $a_1 = -2$, $b_1 = -3$, $a_2 = 1$, $b_2 = 1$
- Saturation constant: M = 3
- Quantizer error: Δ = 0.5
- Sampling delay / dwell-time: $\phi = 0.001$

Fig. 5 (right): Trajectories illustrating local region of attraction Λ and final set of states Ω. Trajectories entering (and staying in) Ω are in green, while those that diverge due to saturation are in red. Blue circles are ellipsoids containing the square equivalence classes defined by the digitizer. Red stars are quantizer values.



Conclusions

- Presented an **algorithmic method using LMIs and reachability of hybrid systems to algorithmically prove stability** of digitally interconnected linear systems.
- Future work to reduce number of LMIs being solved to make the method more scalable.
- Method could be applied to stability verification for some DCPS.

References

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[1] C. Langbort, R. Chandra, R. D'Andrea, "Distributed Control Design for Systems Interconnected over an Arbitrary Graph," *IEEE Transactions on Automatic Control*, Sept. 2004, vol. 49, no. 9.

[2] G. Frehse, "PHAVer: Algorithmic Verification of Hybrid Systems Past HyTech," in *Proceedings of the Fifth International Workshop on Hybrid Systems: Computation and Control (HSCC)*, Springer-Verlag, 2005.