Reachable Set Estimation and Control for Switched Linear Systems with Dwell-Time Restriction

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***Abstract*— The reachable set estimation and control problems for continuous-time switched linear systems are addressed in this paper. First, a general result on reachable set estimation for switched system is proposed based on a Lyapunov function approach. Then, with the help of a class of time-scheduled Lyapunov functions, a numerically tractable sufficient condition ensuring the system state bounded in a prescribed set is derived for switched systems under dwell time constraint. Moreover, a time-scheduled state feedback controller is designed to ensure the state trajectories of the closed-loop system are confined in a prescribed set. Finally, a networked control system subject to packet dropouts is modeled as a switched system with dwell time constraints, and the controller design problem is studied as an application of our results.**

I. Introduction

Switched systems have emerged as an important class of hybrid systems and represent an active area of current research in the field of control systems [1]–[3]. A switched system is composed of a family of continuous or discrete-time subsystems along with a switching rule governing the switching between the subsystems. Generally, the stability and stabilization problems are the main concerns in the field of switched systems. It has been established that Lyapunov function techniques are effective to deal with stability and stabilization problems for switched systems, e.g. see [4]–[8]. Combining multiple Lyapunov function (MLF), the dwell time and average dwell time properties of relatively slowly switched systems have been investigated [9]–[15].

Reachable set estimation aims to derive a closed bounded set that constrains all the state trajectories generated by a dynamic system with a prescribed class of initial state set and inputs. Reachable set estimation is not only of theoretical interest in robust control theory [16], but also closely related to practical engineering for the safety verification problems [17]. In some early work, reachable set bounding was considered in the context of state estimation and it has later received a lot of attention in parameter estimation, see [18] and references therein. Recently, employing ellipsoidal techniques based on Lyapunov function approaches to estimate the reachable sets for different class of systems attracts many researchers' attention. In the framework of bounding ellipsoid, the quadratic Lyapunov function has played a fundamental role in the reachable set estimation problem, and it has been developed to time-delay systems [19]–[21], singular systems [22], discrete-time switched systems [23]. However, according to the best of the authors' knowledge, the reachable set estimation for continuous-time switched systems with constrained switching law, has not been fully investigated, and it motivates our study in this paper.

In this paper, the problems of reachable set estimation and control synthesis for continuous-time switched linear systems will be investigated. First, a general result based on Lyapunov function approach is presented. Then, under the framework of dwell time and with the help of a class of time-scheduled quadratic Lyapunov functions, a linear matrix inequality (LMI) based sufficient condition is proposed to estimate the reachable set. For the control synthesis, a time-scheduled feedback controller is designed to ensure the state trajectories being contained in a prescribed set and, moreover, an optimization problem is formulated to obtain an optimal controller gain to make the reachable set of closed-loop system as small as possible. As an application of our result, the control problem for a networked control system with package dropouts is studied. Based on our derived approach, the controller can be designed with an attempt to constrain state trajectories in a prescribed bounding ellipsoidal region.

Notation: The notations in this paper are fairly standard. $\mathbb{S}_{+}^{n\times n}$ is the set of real symmetric positive definite $n \times n$ matrices. In symmetric block matrices, we use * as an ellipsis for the terms that are introduced by symmetry. $\mathrm{diag}\{\cdots\}$ denotes a block-diagonal matrix and $\mathrm{int}[\cdot]$ rounds the element to the nearest integer towards zero.

II. Preliminaries and Problem Formulation

Let us consider a continuous-time switched linear system in the form of

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\omega,\sigma(t)}\omega(t) + B_{u,\sigma(t)}u(t) \quad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ are the state of the system, and the initial state x_0 is assumed to be settled in a bounded ellipsoid as

$$x_0 \in \mathcal{X}_0 \triangleq \{x_0 \in \mathbb{R}^{n_x} \mid x_0^\top R_0 x_0 \leq 1, R_0 \in \mathbb{S}_{+}^{n_x\times n_x}\} \quad (2)$$

and $\omega(t) \in \mathbb{R}^{n_\omega}$ is the disturbance input vector which is assumed to satisfy the following ellipsoidal constraint

$$\omega(t) \in \mathcal{W} \triangleq \{\omega \in \mathbb{R}^{n_\omega} \mid \omega^\top R_\omega \omega \leq 1, R_\omega \in \mathbb{S}_{+}^{n_\omega\times n_\omega}\} \quad (3)$$

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and $u(t) \in \mathbb{R}^{n_u}$ is the control input to be designed.

Define an index set $\mathcal{M} \triangleq \{1, 2, \ldots, N\}$, where N is the number of modes and, $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{M}$ denotes the switching function, which is assumed to be a piecewise constant function continuous from right. The switching instants are expressed by a sequence $\mathcal{S} \triangleq \{t_k\}_{k\in\mathbb{N}}$, where t_0 denotes the initial time and t_k denotes the kth switching instant. Then, we define $\mathcal{I}_i \triangleq \{t \in \mathbb{R}_{\geq 0} \mid \sigma(t) = i, i \in \mathcal{M}\}$ to denote the activation time interval for ith mode.

The first problem considered in this paper is the reachable set estimation problem for switched system (1) with control input $u(t) = 0$, and the initial state satisfying (2), disturbance input satisfying (3). The reachable set is defined as

$$\mathcal{R}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x(t), x_0, \omega(t) \text{ satisfy } (1), (2), (3)\} \quad (4)$$

Then, the mode-dependent state feedback controller is considered, which has a time-scheduled structure as

$$u(t) = K_{\sigma(t)}(t)x(t) \quad (5)$$

Substituting above controller (5) into system (1), the closed-loop system becomes

$$\dot{x}(t) = \bar{A}_{\sigma(t)}(t)x(t) + B_{\omega,\sigma(t)}\omega(t) \quad (6)$$

where $\bar{A}_{\sigma(t)}(t) = A_{\sigma(t)} + B_{u,\sigma(t)}K_{\sigma(t)}(t)$.

The control objective is to ensure the state trajectory $x(t)$ contained in a given set

$$\tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x^\top R_x x \leq 1, R_x \in \mathbb{S}_+^{n_x \times n_x}\} \quad (7)$$

The above two problems are the main concerns in this paper. In the rest of this paper, the reachable set estimation problem will be studied at first, then based on the reachable set estimation results, the state feedback controller design problem will be addressed.

III. Reachable Set Estimation

A. General Lemma

First, a general lemma is presented to introduce the main idea to determine the over approximate set $\tilde{\mathcal{R}}_x$ for switched system (6), note that switched system (1) with $u(t) = 0$ is a particular case of $\bar{A}_i(t)$ being time-invariant.

Lemma 1: Consider switched system (6) under initial state condition (2) and disturbance input condition (3). If there exist a family of Lyapunov functions $V_i : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, $i \in \mathcal{M}$, satisfying $V_i(0) = 0$ and $V_i(x) > 0$, $\forall x \neq 0$, $\forall i \in \mathcal{M}$, and scalars $\alpha > 0$, $0 < \beta \leq 1$ such that

$$F_i(t) \leq 0, \ \forall t \in \mathcal{I}_i, \forall i \in \mathcal{M} \quad (8)$$

$$G_{i,j}(t_k) \leq 0, \ \forall t_k \in \mathcal{S}, i \neq j, \forall i, j \in \mathcal{M} \quad (9)$$

$$V_i(x_0) \leq x_0^\top R_0 x_0, \ \forall i \in \mathcal{M} \quad (10)$$

where $F_i(t) = \dot{V}_i(x(t)) + \alpha V_i(x(t)) - \alpha\omega^\top(t)R_\omega\omega(t)$ and $G_{i,j}(t_k) = V_i(x(t_k^+)) - \beta V_j(x(t_k^-)) + \beta - 1$. Then, the reachable set $\mathcal{R}_x$ satisfies $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid V_i(x) \leq 1, i \in \mathcal{M}\}$.

Proof: Define the following Lyapunov function as

$$V(t) = \sum\nolimits_{i\in\mathcal{M}} \xi_i(t)V_i(x(t)) \quad (11)$$

where $\xi_i : \mathbb{R}_{\geq 0} \to \{0, 1\}$ and $\sum_{i\in\mathcal{M}} \xi_i(t) = 1$ is the indicator function indicating the active modes at t.

First we consider any $t \in [t_k, t_{k+1}) \subset \mathcal{I}_i$, $\forall i \in \mathcal{M}$. (8) implies $\dot{V}(t) \leq -\alpha V(t) + \alpha\omega^\top(t)R_\omega\omega(t)$, $t \in [t_k, t_{k+1})$. Multiply both sides of this inequality with $e^{\alpha(t-t_k)}$ and then integrating it over $[t_k, t)$, we have $V(t) \leq e^{-\alpha(t-t_k)}V(t_k^+) + \int_{t_k}^t e^{-\alpha(t-s)}\omega^\top(s)R_\omega\omega(s)\mathrm{d}s$. Due to $\omega^\top(t)R_\omega\omega(t) \leq 1$, $\forall t \in \mathbb{R}_{\geq 0}$, we have the following result

$$\begin{aligned} V(t) &\leq e^{-\alpha(t-t_k)}V(t_k^+) + \int_{t_k}^t e^{-\alpha(t-s)}\mathrm{d}s \\ &= e^{-\alpha(t-t_k)}V(t_k^+) + 1 - e^{-\alpha(t-t_k)} \end{aligned} \quad (12)$$

and it can be rewritten to

$$V(t) - 1 \leq e^{-\alpha(t-t_k)}(V(t_k^+) - 1), \ t \in [t_k, t_{k+1}) \quad (13)$$

Next, we consider $t_k \in \mathcal{S}$. From (9), we can obtain $V(t_k^+) \leq \beta V(t_k^-) + 1 - \beta$, $t_k \in \mathcal{S}$, which equals to

$$V(t_k^+) - 1 \leq \beta(V(t_k^-) - 1), \ t_k \in \mathcal{S} \quad (14)$$

Combining (13) and (14), for $\forall t \in \mathbb{R}_{\geq 0}$, it can be obtained $V(t) - 1 \leq \cdots \leq \beta^{\mathrm{Num}(t-t_0)}e^{-\alpha(t-t_0)}(V(t_0) - 1)$, where $\mathrm{Num}(t - t_0)$ is the number of switchings in $[t_0, t)$. Due to $\alpha > 0$ and $0 < \beta \leq 1$, it means that $V(t) - 1 \leq V(t_0) - 1$, $\forall t \in \mathbb{R}_{\geq 0}$. Moreover, (10) implies $V(t_0) \leq x_0^\top R_0 x_0 \leq 1$, and it yields $V(t) \leq 1$, $\forall t \in \mathbb{R}_{\geq 0}$ holds, so $x(t) \in \tilde{\mathcal{R}}_x$, $\forall t \in \mathbb{R}_{\geq 0}$, where $\tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid V_i(x) \leq 1, i \in \mathcal{M}\}$. ∎

Although Lemma 1 provides a general framework to deal with the reachable set estimation problem, it is trivial in actual use, since it does not provide any available computational techniques for the construction of Lyapunov functions $V_i(x(t))$, $i \in \mathcal{M}$ and moreover, the proposed condition (9) requires us to check the values of Lyapunov functions at every the switching instant $t_k \in \mathcal{S}$. However, the switching instant sequence $\mathcal{S}$ usually cannot be specified in advance, and it is impossible to check Lemma 1 for all switching instants t_k in the case of $k \to \infty$.

B. Time-Scheduled Multiple Lyapunov Functions

Based on Lemma 1, we particularly consider a class of switched system with dwell-time constraint.

Definition 1: Given a switching signal function $\sigma(t)$ with a generated switching sequence $\mathcal{S}$, $\tau_{\min} = \inf_{k\in\mathbb{N}}\{t_{k+1} - t_k\}$ is called the minimum dwell time of $\sigma(t)$. $\mathcal{D}_{\tau_{\min}} \triangleq \{\sigma \mid \sigma : \mathbb{R}_{\geq 0} \to \mathcal{M}, t_{k+1} - t_k \geq \tau_{\min}, \forall k \in \mathbb{N}\}$ denotes the set of all switching policies with dwell time greater than $\tau_{\min}$.

Then, inspired by [11], [12], [15], we consider a class of time-scheduled multiple Lyapunov functions as follow

$$V_i(x(t)) = x^\top(t)P_i(t)x(t), \ t \in \mathbb{R}_{\geq 0}, \ i \in \mathcal{M} \quad (15)$$

where $P_i(t) \in \mathbb{S}_+^{n_x \times n_x}$, $i \in \mathcal{M}$ have the following structure:

Consider the interval $[t_k, t_k + \tau_{\min})$, we divide it into L segments described as $\mathcal{L}_{k,q} \triangleq [t_k + \theta_q, t_k + \theta_{q+1})$, $q = 0, 1, \ldots, L-1$ of equal lengths $h = \tau_{\min}/L$, and then $\theta_0 = 0$ and $\theta_q = qh = q\tau_{\min}/L$. We consider a class of continuous matrix function $P_i(t)$, $t \in [t_k, t_k + \tau_{\min})$ chosen to be linear within each segments $\mathcal{L}_{k,q}$, $q = 0, 1, \ldots, L - 1$. Explicitly, we can see that $\bigcup_{n=0}^{L-1}\mathcal{L}_{k,n} = [t_k, t_k+\tau_{\min})$ and $\mathcal{L}_{k,n}\cap\mathcal{L}_{k,m}=\emptyset$, $n\neq m$. Letting $P_{i,q}=P_i(t_k+\theta_q)$, then since the matrix function $P_i(t)$ is piecewise linear in $[t_k, t_k+\tau_{\min})$, it can be expressed in terms of the values at dividing points using a linear interpolation formula, that is, for $0\leq\mu\leq 1$, $q=0,1,\ldots,L-1$,

$$P_i(t)=P_i(\mu)=(1-\mu)P_{i,q}+\mu P_{i,q+1},\ t\in\mathcal{L}_{k,q},\ i\in\mathcal{M} \tag{16}$$

where $\mu=L(t-t_k-\theta_q)/\tau_{\min}$.

As a result, the continuous matrix function $P_i(t)\in\mathbb{S}_{+}^{n_x\times n_x}$, $i\in\mathcal{M}$ can be completely determined by $P_{i,q}\in\mathbb{S}_{+}^{n_x\times n_x}$, $q=0,1,\ldots,L$, $i\in\mathcal{M}$, in interval $[t_k, t_k+\tau_{\min})$.

Then, due to $[t_k, t_k+\tau_{\min})\subseteq[t_k, t_{k+1})$, for the remaining time in $[t_k, t_{k+1})$ denoted by $\mathcal{L}_{k,L}\triangleq[t_{k,\min}, t_{k+1})$, $P_i(t)$, $i\in\mathcal{M}$ is set to be

$$P_i(t)=P_{i,L},\ t\in\mathcal{L}_{k,L},\ i\in\mathcal{M} \tag{17}$$

In summary, the $P_i(t)$, $i\in\mathcal{M}$ in Lyapunov function in (15) is defined as

$$P_i(t)=\begin{cases} P_i(\mu), & t\in\mathcal{L}_{k,q},\ q=0,1,\ldots,L-1\\ P_{i,L}, & t\in\mathcal{L}_{k,L}\end{cases} \tag{18}$$

where μ is defined in (16).

C. Reachable Set Estimation under Dwell Time Constraint

Now, we are ready to propose out main result as follows.

Theorem 1: Given dwell time $\tau_{\min}>0$ and consider switched system (1) with $\sigma(t)\in\mathcal{D}_{\tau_{\min}}$ under initial state condition (2), disturbance input condition (3) and $u(t)=0$. If there exist a set of matrices $P_{i,q}\in\mathbb{S}_{+}^{n_x\times n_x}$, $q=0,1,\ldots,L$, $i\in\mathcal{M}$ and a scalar $\alpha>0$ such that for $\forall i,j\in\mathcal{M}$

$$\begin{bmatrix}\Xi_{i,q}+\Psi_{i,q} & *\\ B_{\omega,i}^{\top}P_{i,q} & -\alpha R_{\omega}\end{bmatrix}\prec 0,\ q=0,\ldots,L-1 \tag{19}$$

$$\begin{bmatrix}\Xi_{i,q+1}+\Psi_{i,q} & *\\ B_{\omega,i}^{\top}P_{i,q} & -\alpha R_{\omega}\end{bmatrix}\prec 0,\ q=0,\ldots,L-1 \tag{20}$$

$$\begin{bmatrix}\Xi_{i,L} & *\\ B_{\omega,i}^{\top}P_{i,L} & -\alpha R_{\omega}\end{bmatrix}\prec 0 \tag{21}$$

$$P_{i,0}-P_{j,L}\prec 0,\ i\neq j \tag{22}$$

$$P_{i,0}-R_0\prec 0 \tag{23}$$

where $\Xi_{i,q}=A_i^{\top}P_{i,q}+P_{i,q}A_i+\alpha P_{i,q}$ and $\Psi_{i,q}=L(P_{i,q+1}-P_{i,q})/\tau_{\min}$. Then, the reachable set $\mathcal{R}_x\subseteq\mathcal{R}_x\triangleq\{x\in\mathbb{R}^{n_x}\mid x^{\top}P_{i,q}x\leq 1, q=0,1,\ldots,L, i\in\mathcal{M}\}$.

Proof: Construct Lyapunov function as

$$V(t)=\sum\nolimits_{i\in\mathcal{M}}\xi_i(t)x^{\top}(t)P_i(t)x(t) \tag{24}$$

where $P_i(t)$, $i\in\mathcal{M}$, is defined by (18) and $\xi_i(\cdot)$ is defined same as (11).

First, let us consider $F_i(t)=\dot{V}(t)+\alpha V(t)-\alpha\omega^{\top}(t)R_{\omega}\omega(t)$, which can be rewritten to

$$F_i(t)=\chi^{\top}(t)\begin{bmatrix}\Xi_i(t)+\dot{P}_i(t) & *\\ B_{\omega,i}^{\top}P_i(t) & -\alpha R_{\omega}\end{bmatrix}\chi(t) \tag{25}$$

where $\chi^{\top}(t)=[x^{\top}(t)\quad\omega^{\top}(t)]$ and $\Xi_i(t)=A_i^{\top}P_i(t)+P_i(t)A_i+\alpha P_i(t)$.

TABLE I

COMPUTATIONAL COMPLEXITIES OF THEOREM 1 WITH A FIXED α

Number of Decision Variables	LMI Constraints Size
$nN(L+1)(n+1)/2$	$2nN(N+2L+1)$

Suppose $\sigma(t)=i$, $t\in\mathcal{L}_{k,q}$, $q=0,\ldots,L-1$, one has

$$\begin{bmatrix}\Xi_i(t)+\dot{P}_i(t) & *\\ B_{\omega,i}^{\top}P_i(t) & -\alpha R_{\omega}\end{bmatrix}=(1-\mu)\Pi_{i,1}+\mu\Pi_{i,2} \tag{26}$$

where $\Pi_{i,1}=\begin{bmatrix}\Xi_{i,q}+\Psi_{i,q} & *\\ B_{\omega,i}^{\top}P_{i,q} & -\alpha R_{\omega}\end{bmatrix}$ and $\Pi_{i,2}=\begin{bmatrix}\Xi_{i,q+1}+\Psi_{i,q+1} & *\\ B_{\omega,i}^{\top}P_{i,q+1} & -\alpha R_{\omega}\end{bmatrix}$.

Furthermore, we can see $\dot{P}_i(t)=(P_{i,q+1}-P_{i,q})\dot{\mu}$, $t\in\mathcal{L}_{k,q}$, $q=0,\ldots,L-1$, and because of $\mu=L(t-t_k-\theta_q)/\tau_{\min}$, it implies $\dot{\mu}=L/\tau_{\min}$, leading to $\dot{P}_i(t)=\Psi_{i,q}$, $t\in\mathcal{L}_{k,q}$, $q=0,\ldots,L-1$,. By (19), (20), it leads to

$$F_i(t)<0,\ \forall t\in\bigcup\nolimits_{n=0}^{L-1}\mathcal{L}_{k,n}=[t_k, t_k+\tau_{\min}) \tag{27}$$

Then, we consider $t\in\mathcal{L}_{k,L}$. Since $P_i(t)=P_{i,L}$, $t\in\mathcal{L}_{k,L}$, we have $\dot{P}_i(t)=0$, $\forall t\in\mathcal{L}_{k,L}$, thus (21) guarantees that $F_i(t)<0$, $\forall t\in\mathcal{L}_{k,L}$. Together with (27), we can conclude that $F_i(t)<0$, $\forall t\in\mathcal{I}_i$, $\forall i\in\mathcal{M}$, which means (8) in Lemma 1 holds.

Next, (22) ensures (9) holds with $\beta=1$ and (23) guarantees (10) holds. Therefore, we have the reachable set $\mathcal{R}_x\subseteq\tilde{\mathcal{R}}_x\triangleq\{x\in\mathbb{R}^{n_x}\mid x^{\top}P_{i,q}x\leq 1, q=0,1,\ldots,L, i\in\mathcal{M}\}$ by Lemma 1. ■

Remark 1: Parameter L implies the number of segments consisting of the dwell time interval $[t_k, t_k+\tau_{\min})$. A larger L yields a finer division of $[t_k, t_k+\tau_{\min})$, and a less conservative result can be consequently obtained, which will be demonstrated by a numerical example later. However, the computational cost increases as L grows, since a larger L inevitably introduces more decision variables and LMI constraints, see TABLE I for the computational complexity analysis for Theorem 1.

The set $\tilde{\mathcal{R}}_x$ is usually expected to be as small as possible to achieve a precise estimation of reachable set $\mathcal{R}_x$. Based on Theorem 2, one may add an additional constraint that

$$P_{i,q}\succeq\epsilon I,\ \epsilon>0,\ \forall q=0,1,\ldots,L,\ \forall i\in\mathcal{M} \tag{28}$$

which implies that $\epsilon x^{\top}(t)x(t)\leq x^{\top}(t)P_{i,q}x(t)\leq 1$, namely $x(t)\in\mathcal{B}(0,1/\sqrt{\epsilon})\triangleq\{x\in\mathbb{R}^n\mid\|x\|\leq 1/\sqrt{\epsilon}\}$, $\forall t\in\mathbb{R}_{\geq 0}$, so we have to maximize ϵ to obtain a smallest reachable set with respect to ϵ. Given an L, the smallest ball $\mathcal{B}(0,1/\sqrt{\epsilon})\triangleq\{x\in\mathbb{R}^n\mid\|x\|<1/\sqrt{\epsilon}\}$ containing the trajectories of state $x(t)$ in the framework of our approach can be obtained. Based on Theorem 1, an optimization problem can be formulated by adding (28) with (19)–(23) as follows

$$\max\ \epsilon\ \text{s.t. (28) and (19)}-\text{(23)} \tag{29}$$

In the extreme case with $L=0$, $P_{i,q}$, shrinks to P_i, moreover, due to (35), we have to choose $P_i=P_j$, $i\neq j$. Thus, Theorem 2 is reduced to the following corollary.

Corollary 1: Consider switched system (1) under initial state condition (2), disturbance input condition (3) and $u(t) = 0$. If there exist a matrix $P \in \mathbb{S}_+^{n_x \times n_x}$ and a scalar $\alpha > 0$ such that

$$\begin{bmatrix} A_i^\top P + PA_i + \alpha P & * \\ B_{\omega,i}^\top P & -\alpha R_\omega \end{bmatrix} \prec 0,\ \forall i \in \mathcal{M} \quad (30)$$

$$P - R_0 \prec 0 \quad (31)$$

Then, the reachable set $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x^\top Px \leq 1\}$.

Remark 2: Corollary 1 is actually the straightforward result derived based on the well-known common Lyapunov function approach. It can be observed that there is no restriction for the dwell time, this means that it can be used for arbitrary switching case which includes broader classes of switching signals, however, the cost is the increase of conservativeness of the estimation results.

IV. Time-Scheduled Feedback Controller Design

In this section, the controller design problem will be considered in the framework of dwell time. Based on Theorem 1, the following result can be derived for controller design.

Theorem 2: Given dwell time $\tau_{\min} > 0$ and consider switched system (1) with $\sigma(t) \in \mathcal{D}_{\tau_{\min}}$ under initial state condition (2) and disturbance $\omega(t)$ satisfying (3). If there exist a set of matrices $S_{i,q} \in \mathbb{S}_+^{n_x \times n_x}$, $X_{i,q} \in \mathbb{R}^{n_u \times n_x}$, $q = 0, 1, \ldots, L$, $i \in \mathcal{M}$ and a scalar $\alpha > 0$ such that for $\forall i, j \in \mathcal{M}$

$$\begin{bmatrix} \Xi_{i,q} - \Psi_{i,q} & * \\ B_{\omega,i}^\top & -\alpha R_\omega \end{bmatrix} \prec 0,\ q = 0, \ldots, L-1 \quad (32)$$

$$\begin{bmatrix} \Xi_{i,q+1} - \Psi_{i,q} & * \\ B_{\omega,i}^\top & -\alpha R_\omega \end{bmatrix} \prec 0,\ q = 0, \ldots, L-1 \quad (33)$$

$$\begin{bmatrix} \Xi_{i,L} & * \\ B_{\omega,i}^\top & -\alpha R_\omega \end{bmatrix} \prec 0 \quad (34)$$

$$S_{j,L} - S_{i,0} \prec 0,\ i \neq j \quad (35)$$

$$R_0^{-1} - S_{i,0} \prec 0 \quad (36)$$

$$S_{i,q} - R_x^{-1} \prec 0,\ q = 0, \ldots, L-1 \quad (37)$$

where $\Xi_{i,q} = A_i S_{i,q} + S_{i,q} A_i^\top + B_{i,u} X_{i,q} + X_{i,q}^\top B_{i,u}^\top + \alpha S_{i,q}$ and $\Psi_{i,q} = L(S_{i,q+1} - S_{i,q})/\tau_{\min}$. Then, the closed-loop system (6) with controller gain $K_i(t) = X_i(t) S_i^{-1}(t)$ has a reachable set $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x^\top R_x x \leq 1\}$, where $S_i(t)$ and $X_i(t)$ are given by

$$S_i(t) = \begin{cases} (1-\mu) S_{i,q} + \mu S_{i,q+1} & t \in \mathcal{L}_{k,q} \\ S_{i,L} & t \in \mathcal{L}_{k,L} \end{cases} \quad (38)$$

$$X_i(t) = \begin{cases} (1-\mu) X_{i,q} + \mu X_{i,q} & t \in \mathcal{L}_{k,q} \\ X_{i,L} & t \in \mathcal{L}_{k,L} \end{cases} \quad (39)$$

where $\mu = L(t - t_k)/\tau_{\min} - q$ and q is determined by

$$q = \begin{cases} \mathrm{int}[L(t - t_k)/\tau_{\min}] & 0 \leq m < L \\ L & q \geq L \end{cases} \quad (40)$$

Proof: Since $S_{i,q} \succ 0$, it implies $S_i(t)$ defined by (60) is positive definite, and thus we have $S_i^{-1}(t) \succ 0$. Then, a Lyapunov function for closed-loop system (6) can be constructed as follows:

$$V(t) = \sum\nolimits_{i \in \mathcal{M}} \xi_i(t) x^\top(t) S_i^{-1}(t) x(t) \quad (41)$$

where $\xi_i(\cdot)$ is defined same as (11).

Substituting $X_i(t) = K_i(t) S_i(t)$, (32)–(34) ensure the following inequality holds

$$\begin{bmatrix} \bar{A}_i(t) S_i(t) + S_i(t) \bar{A}_i^\top(t) + \alpha S_i(t) - \dot{S}_i(t) & * \\ B_{\omega,i}^\top & -\alpha R_\omega \end{bmatrix} \prec 0 \quad (42)$$

Multiplying both side of (42) by $\mathrm{diag}\{S_i^{-1}(t), I\}$ and using $\dot{S}_i^{-1}(t) = -S_i^{-1}(t) \dot{S}_i(t) S_i^{-1}(t)$, we have

$$\begin{bmatrix} \Xi_i(t) & * \\ B_{\omega,i}^\top S_i^{-1}(t) & -\alpha R_\omega \end{bmatrix} \prec 0 \quad (43)$$

where $\Xi_i(t) = \bar{A}_i^\top(t) S_i^{-1}(t) + S_i^{-1}(t) \bar{A}_i(t) + \dot{S}_i^{-1}(t) + \alpha S_i^{-1}(t)$. It implies (8) in Lemma 1 holds.

Then, we consider (35) and (36). If (35) holds, it equals to $\Phi = \begin{bmatrix} -S_{j,L}^{-1} & I \\ I & -S_{i,0} \end{bmatrix} \prec 0$ by Schur complement. Then, further considering the Schur complement of Φ, we obtain $S_{i,0}^{-1} - S_{j,L}^{-1} \prec 0$ implying (9) in Lemma 1 holds with $\beta = 1$. Similarly, (10) can be guaranteed by (36). Thus, we have the reachable set $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x^\top S_{i,q}^{-1} x \leq 1, q = 0, 1, \ldots, L, i \in \mathcal{M}\}$ by Lemma 1. Finally, from (37), we have $x^\top R_x x < x^\top S_{i,q}^{-1} x \leq 1$, $q = 0, 1, \ldots, L$, $i \in \mathcal{M}$, which implies $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x^\top R_x x \leq 1\}$. ■

Remark 3: In order to obtain a optimized controller for the smallest reachable set estimation for closed loop system, we can add the following constraint

$$S_{i,q} - \delta I \prec 0,\ \delta > 0, q = 0, \ldots, L, i \in \mathcal{M} \quad (44)$$

The above inequality ensures the $x(t) \in \mathcal{B}(0, \sqrt{\delta}) \triangleq \{x \in \mathbb{R}^n \mid \|x\| \leq \sqrt{\delta}\}$. Given an L, the smallest ball $\mathcal{B}(0, \sqrt{\delta})$ containing the reachable set $\tilde{\mathcal{R}}_x$ can be obtained by the following optimization problem

$$\min \delta \text{ s.t. } (44) \text{ and } (32) - (36) \quad (45)$$

Corollary 2: Consider switched system (1) under initial state condition (2) and disturbance $\omega(t)$ satisfying (3). If there exist matrices $S \in \mathbb{S}_+^{n_x \times n_x}$, $X_i \in \mathbb{R}_+^{n_\omega \times n_x}$, $i \in \mathcal{M}$ and a scalar $\alpha > 0$ such that

$$\begin{bmatrix} \Xi_i & * \\ B_{\omega,i}^\top & -\alpha R_\omega \end{bmatrix} \prec 0,\ \forall i \in \mathcal{M} \quad (46)$$

$$R_0^{-1} \prec S \prec R_x^{-1} \quad (47)$$

where $\Xi_{i,q} = A_i S + S A_i^\top + B_{u,i} X_i + X_i^\top B_{u,i}^\top + \alpha S$. Then, the closed-loop system (6) with controller gain $K_i = X_i S$ has a reachable set $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x^\top R_x x \leq 1\}$.

Proof: It can be easily proved by letting $L = 0$ in Theorem 2, so the proof is omitted here. ■

Though Corollary 2 provides constant feedback gains K_i, $i \in \mathcal{M}$ which does not require online computation as $K_i(t)$, $i \in \mathcal{M}$ do. This feature is more convenient for controller realization in practice, but the conservatism grows in comparison with the case of $L > 0$.

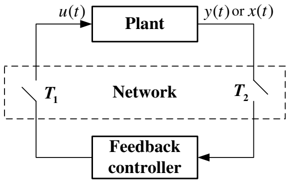

Fig. 1. Packet dropouts in networked control system

V. Application in Networked Control Systems

Consider a networked control system with packet dropouts in both forward and backward channels, where the packet dropouts can be modeled as switches open behavior, which is illustrated in Fig. 1. When T_1 is closed, the controller output is successfully transmitted to the actuator; whereas when it is open, the output of the switch becomes zero and a packet is lost, and we have in this case $u(t) = 0$. The situation is the same for the backward channel. In absence of packet dropouts, the state feedback controller works well during interval $\Gamma_{1,k} \triangleq [t_{2k}, t_{2k+1}), k \in \mathbb{N}$. However, due to the occurrence of packet dropouts, the controller is considered to be not available, namely $u(t) = 0$, in the time interval $\Gamma_{2,k} \triangleq [t_{2k+1}, t_{2k+2}), k \in \mathbb{N}$.

Assumption 1: The following assumptions are made:

1) There exists a uniform lower-bound $\tau_{\min}$ on the lengths of $\Gamma_{1,k}$, $k \in \mathbb{N}$, that is $t_{2k+1} - t_{2k} \geq \tau_{\min}$, $\forall k \in \mathbb{N}$.
2) There exist a uniform upper-bound $\psi_{\max}$ on the lengths of $\Gamma_{2,k}$, $k \in \mathbb{N}$, that is $t_{2k+2} - t_{2k+1} \leq \psi_{\max}$, $\forall k \in \mathbb{N}$.

The plant we consider is a linear system

$$\dot{x}(t) = Ax(t) + B_\omega \omega(t) + B_u u(t) \tag{48}$$

and the controller is considered to be $u(t) = K(t)x(t),\ t \in \Gamma_{1,k}$. In summary, the networked control system with packet dropouts can be described as follows

$$\dot{x}(t) = A_{\sigma(t)}(t)x(t) + B_\omega \omega(t) \tag{49}$$

where $A_1(t) = A + B_u K(t)$ and $A_2(t) = A$, and the switching function $\sigma(t)$ is

$$\sigma(t) = \begin{cases} 1 & t \in \Gamma_{1,k} \\ 2 & t \in \Gamma_{2,k} \end{cases} \tag{50}$$

Theorem 3: Under Assumption 1 and consider networked control system (48) under initial state condition (2) and disturbance $\omega(t)$ satisfying (3). If there exist a set of matrices $S_{i,q} \in \mathbb{S}_+^{n_x \times n_x}$, $X_{i,q} \in \mathbb{R}_+^{n_u \times n_x}$, $q = 0, 1, \ldots, L$, $i = 1, 2$ and a scalar $\alpha > 0$ such that

$$\begin{bmatrix} \Xi_{1,q} - \Psi_{1,q} & * \\ B_\omega^\top & -\alpha R_\omega \end{bmatrix} \prec 0,\ q = 0, \ldots, L-1 \tag{51}$$

$$\begin{bmatrix} \Xi_{1,q+1} - \Psi_{1,q} & * \\ B_\omega^\top & -\alpha R_\omega \end{bmatrix} \prec 0,\ q = 0, \ldots, L-1 \tag{52}$$

$$\begin{bmatrix} \Xi_{1,L} & * \\ B_\omega^\top & -\alpha R_\omega \end{bmatrix} \prec 0 \tag{53}$$

$$\begin{bmatrix} \Xi_{2,q} - \Psi_{2,q} & * \\ B_\omega^\top & -\alpha R_\omega \end{bmatrix} \prec 0,\ q = 0, \ldots, L-1 \tag{54}$$

$$\begin{bmatrix} \Xi_{2,q+1} - \Psi_{2,q} & * \\ B_\omega^\top & -\alpha R_\omega \end{bmatrix} \prec 0,\ q = 0, \ldots, L-1 \tag{55}$$

$$S_{1,L} - S_{2,0} \prec 0 \tag{56}$$

$$S_{2,q} - S_{1,0} \prec 0,\ q = 0, \ldots, L \tag{57}$$

$$R_0^{-1} - S_{i,0} \prec 0,\ i = 1, 2 \tag{58}$$

$$S_{i,q} - R_x^{-1} \prec 0,\ i = 1, 2,\ q = 0, \ldots, L \tag{59}$$

where $\Xi_{1,q} = AS_{1,q} + S_{1,q}A^\top + B_u X_{1,q} + X_{1,q}^\top B_u^\top + \alpha S_{1,q}$, $\Xi_{2,q} = AS_{i,q} + S_{2,q}A^\top + \alpha S_{2,q}$ and $\Psi_{1,q} = L(S_{1,q+1} - S_{1,q})/\tau_{\min}$, $\Psi_{2,q} = L(S_{2,q+1} - S_{2,q})/\psi_{\max}$. Then, the closed-loop system (49) with controller gain $K(t) = X_1(t)S_1^{-1}(t)$, $t \in \Gamma_{1,k}$ has a reachable set $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x^\top R_x x \leq 1\}$, where $S_1(t)$ and $X_1(t)$ are

$$S_1(t) = \begin{cases} (1-\mu)S_{1,q} + \mu S_{1,q+1} & t \in \mathcal{L}_{k,q} \\ S_{1,L} & t \in \mathcal{L}_{k,L} \end{cases} \tag{60}$$

$$X_1(t) = \begin{cases} (1-\mu)X_{1,q} + \mu X_{1,q} & t \in \mathcal{L}_{k,q} \\ X_{1,L} & t \in \mathcal{L}_{k,L} \end{cases} \tag{61}$$

where $\mu = L(t - t_{2k})/\tau_{\min} - q$ and q is determined by

$$q = \begin{cases} \mathrm{int}[L(t - t_{2k})/\tau_{\min}] & 0 \leq m < L \\ L & q \geq L \end{cases} \tag{62}$$

Proof: By the similar guidelines in Theorem 2, conditions (51), (52) and (53) ensures that (8) in Lemma 1 holds for interval $\Gamma_{1,k}$, and (54), (55) guarantee (8) in Lemma 1 holds for $\Gamma_{2,k}$. Then, (56) and (57) implies (9) holds for switching instants t_{2k+1}, t_{2k}, respectively. Finally, (10) can be guaranteed by (58). Thus, according Lemma 1, the reachable set is obtained as $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x^\top S_{i,q}^{-1} x \leq 1, q = 0, 1, \ldots, L, i = 1, 2\}$. Using (59), we have $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x^\top R_x x \leq 1\}$. ■

By adding the following constraint

$$S_{i,q} - \delta I \prec 0,\ \delta > 0,\ q = 0, \ldots, L,\ i = 1, 2 \tag{63}$$

The smallest ball $\mathcal{B}(0, \sqrt{\delta})$ containing the reachable set $\tilde{\mathcal{R}}_x$ can be obtained by the following optimization problem

$$\min \delta \text{ s.t. } (63) \text{ and } (51) - (58) \tag{64}$$

Example 1: Consider the plant described by

$$A = \begin{bmatrix} 1.5 & 2.5 \\ 1.5 & 1.2 \end{bmatrix},\ B_u = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix},\ B_\omega = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The initial state is assumed to satisfy $x_0 \in \{x_0 \in \mathbb{R}^n \mid \|x_0\| \leq 1\}$, and the control objective is to ensure the state trajectories satisfies $x(t) \in \{x \in \mathbb{R}^n \mid \|x\| \leq 2\}$. Assume that the minimal reliable time for a group of successfully transmitted information is $\tau_{\min} = 0.5$ second, and the maximal time for a group of successive packet dropouts is $\psi_{\max} = 0.1$ second. Let $\alpha = 0$ due to $B_\omega = [0\ \ 0]^\top$, and we can find feasible solution to LMIs (51)–(59) with $L = 1$.

Given an initial state $x_0 = [0.6\ \ 0.8]^\top$, the state response is illustrated in Fig. 2, it can be observed that the state trajectory satisfies $x(t) \in \{x \in \mathbb{R}^n \mid \|x\| \leq 2\}$. Moreover, we generate 500 random state trajectories with random packet dropouts whose lengths are less than 0.1 second, it can be seen that all the trajectories are in the prescribed ball $\mathcal{B}(0, 2)$, which are shown in Fig. 3.

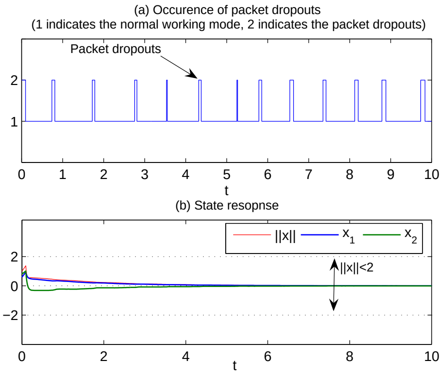

Fig. 2. State response of networked control system with packet dropouts

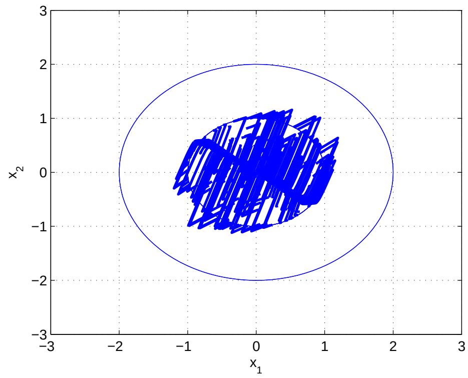

Fig. 3. 500 random state trajectories with random packet dropouts. All the trajectories $x(t)$ generated from the $\|x_0\| \leq 1$ are bounded by $\|x(t)\| \leq 2$.

Finally, in order the show how parameter L works for the controller design, different L are selected for optimization problem (64). From $L = 1$ to $L = 5$, the smallest δ are computed, which are shown in Table II. In Table II, we can see that δ monotonically decreases as L increases, this is consistent with Remark 1. However, a selection of larger L has to afford more computational cost, the computation time grows as L increases in Table II.

TABLE II
δ AND COMPUTATION TIME (C.T.) WITH $L = 1, 2, 3, 4, 5$

	$L = 1$	$L = 2$	$L = 3$	$L = 4$	$L = 5$
δ	1.8795	1.5075	1.4615	1.4434	1.4334
C.T.	0.296 s	0.433 s	0.561 s	0.734 s	0.905 s

VI. Conclusions

By employing a class of time-scheduled Lyapunov functions, the reachable estimation and control problems for switched linear systems under dwell time constraint are investigated in this paper. A sufficient condition has been proposed to estimate the reachable set of switched system by bounding ellipsoids, then based on the estimation result, a time-scheduled state feedback controller gains are obtained, which can ensure the state trajectories of closed-loop system in a prescribed set. Finally, the controller design result is applied into the networked control system with packet dropouts.

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