On Reachable Set Estimation for Discrete-Time Switched Linear Systems under Arbitrary Switching

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Abstract— **This paper addresses the problem of reachable set estimation for discrete-time switched systems under arbitrary switching. By introducing a novel conception called M-step sequence which is capable of characterizing all possible subsystem activation orders during M discrete-time steps, a Lyapunov function based approach is proposed to derive a set of bounding ellipsoids to estimate the reachable set. The proposed approach can cover the previous switched Lyapunov function approach and yields less conservativeness. Moreover, it can be shown that the M-step sequence method can also reduce the conservativeness in stability analysis for discrete-time switched systems under arbitrary switching in contrast to switched Lyapunov function method. Several numerical examples are provided to illustrate our approach.**

I. Introduction

A switched system is composed of a family of continuous or discrete-time subsystems, described by differential or difference equations, respectively, along with a switching rule governing the switching between the subsystems. The motivation for studying such switched systems comes from the fact that switched system can be efficiently used to model many practical systems that are inherently multi-model, thus several dynamical subsystem models are required to describe their behavior. For example, several real-world cyber-physical systems and industrial processes exhibit switching and hybrid nature intrinsically. Generally, the stability and stabilization problems are the main concerns in the field of switched systems, e.g., see [1]–[4] and the references cited therein. One can study the stability and other properties of switched systems with a given the switching rule as a prescribed state space partitioning [5]–[7] or with some known constraints on switching sequence such as dwell time [8] or average dwell time [9] restrictions. For instance, combining multiple Lyapunov function (MLF), the dwell time and average dwell time properties of relatively slowly switched systems have been investigated in the corresponding switched systems [10]–[16]. However, in a number of practical switched systems, the switching sequence is not known *a prior* and these properties have to be examined under arbitrary switching.

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Reachable set estimation aims to derive a closed bounded set that contains all the state trajectories generated by a dynamic system with a prescribed class of initial state set and inputs. Reachable set estimation is not only of theoretical interest in robust control theory [17], but also closely related to practical engineering for the safety verification problems [18]. For example, a dynamic system is regarded to be safe if its reachable set does not intersect with the unsafe or undesirable sets of states. In some early work, reachable set bounding was considered in the context of state estimation and it has later received a lot of attention in parameter estimation, see [19] and references therein. Recently, employing ellipsoidal techniques based on Lyapunov function approaches to estimate the reachable sets for different class of systems attracts many researchers' attention. In the framework of bounding ellipsoid, the quadratic Lyapunov function has played a fundamental role in the reachable set estimation problem, and it has been further extended and developed to time-delay systems [20]–[22], singular systems [23].

For the reachable set estimation problem for discrete-time switched system under arbitrary switching, [24] proposes a method based on switched Lyapunov function approach, and the trajectories are estimated by a set of bounding ellipsoids. The main aim in this paper is to further develop the Lyapunov function approach and reduce its conservativeness in reachable set estimation for discrete-time switched system under arbitrary switching. By introducing the conception of M-step sequence which is able to characterize all possible subsystem activation orders during M steps, an improved method will be proposed in this paper. It should be stressed that the approach in [24] can be recovered by particularly letting $M = 1$ and thus has less conservativeness. Additionally, also in virtue of the advantages of M-step sequence, the less conservativeness emerges in stability analysis for discrete-time switched system in comparison with the well-known switched Lyapuonv function. Finally, several numerical examples are given in order to emphasize the less conservativeness and effectiveness of the approach.

The remainder of this paper is organized as follows: Preliminaries and problem formulation are given in Section II. The main results including the M-step sequence, reachable set estimation and discussion on stability are given in Section III. Numerical examples are provided in Section IV. Conclusions are given in Section V.

Notation: $\mathbb{N}$ represents the set of natural numbers. $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the fields of real numbers and nonnegative real numbers, respectively. $\mathbb{R}^n$ is the vector space of all n-tuples of real numbers, $\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices with real entries. The notation $P \succ 0$ ($P \prec 0$) means P is real symmetric and positive definite (negative definite). $A^\top$ denotes the transpose of A. In symmetric block matrices, we use * as an ellipsis for the terms that are introduced by symmetry. $\mathrm{diag}\{\cdots\}$ denotes a block-diagonal matrix. $|\cdot|$ stands for the Euclidean norm. The bounding ellipsoid is expressed by $\mathcal{E}(R) \triangleq \{x \in \mathbb{R}^n \mid x^\top R x \leq 1, 0 \prec R \in \mathbb{R}^{n\times n}\}$, and ball $\mathcal{B}(x_0, \delta) \triangleq \{x \in \mathbb{R}^n \mid |x - x_0| \leq \delta, x_0 \in \mathbb{R}^n, \delta > 0\}$.

II. Preliminaries and Problem Formulation

In this paper, we consider a class of discrete-time switched linear system in the following form

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}\omega(k) \tag{1}$$

where $x(k), x_0 \in \mathbb{R}^{n_x}$ are the state of the system and initial state, respectively. The switching signal σ is defined as $\sigma : \mathbb{N} \to \mathcal{I}[1, N]$, where N is the number of subsystems involved in the switched system. In this paper, no specific restriction is imposed on switching signal σ, namely the arbitrary switching law is considered in the rest of paper. A_i, and B_i, $i \in \mathcal{I}[1, N]$ are constant system matrices with appropriate dimensions. $\omega(k) \in \mathbb{R}^{n_\omega}$ is the bounded peak input vector which is assumed to satisfy the following constraint

$$\omega(k) \in \mathcal{W} \triangleq \{\omega \in \mathbb{R}^{n_\omega} \mid \omega^\top \omega \leq d^2, d > 0\} \tag{2}$$

The main problem considered in this paper is the reachable set estimation problem for switched system (1) with input $\omega(k)$ satisfying (2). The reachable set is defined as

$$\mathcal{R}_x \triangleq \{x \in \mathbb{R}^{n_x} \mid x_0 = 0, x(k), \omega(k) \text{ satisfy } (1), (2)\} \tag{3}$$

Due to the complex characteristic of switched systems, the accurate reachable set for switched system (1) is hard to compute. The reachable set estimation problem is formulated as follows.

Problem 1: For switched system (1), determine an over approximate set $\tilde{\mathcal{R}}_x$ such that $\mathcal{R}_x \subseteq \tilde{\mathcal{R}}_x$, and the set $\tilde{\mathcal{R}}_x$ should be optimized as small as possible.

The recent solution to compute an over approximate set $\tilde{\mathcal{R}}_x$ is proposed in [24], which is based on switched Lyapunov function approach [25].

Lemma 1: [24] Consider system (1) with input (2). If there exist a set of a family of functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$ satisfying $V_i(0) = 0$ and $V_i(x) > 0$, $\forall x = 0$, $i \in \mathcal{I}[1, N]$, and exist scalars $0 < \alpha_{i,j} < 1$ such that $\forall (i, j) \in \mathcal{I}[1, N] \times \mathcal{I}[1, N]$,

$$V_j(x(k+1)) - \alpha_{i,j}V_i(x(k)) - \frac{1 - \alpha_{i,j}}{d^2}\omega^\top(k)\omega(k) \leq 0 \tag{4}$$

then system (1) is globally uniformly asymptotically stable and we have $\exists i \in \mathcal{I}[1, N]$ such that $V_i(x(k)) \leq 1$ for all $x(0)$ satisfying $V_i(x(0)) \leq 1$, $\forall i \in \mathcal{I}[1, N]$.

In the framework of quadratic switched Lyapunov function, the following result for reachable set estimation stems from above Lemma.

Lemma 2: [24] Consider system (1) with input (2). If there exist matrices $P_i \succ 0$, $i \in \mathcal{I}[1, N]$ and scalars $0 < \alpha_{i,j} < 1$ such that $\forall (i, j) \in \mathcal{I}[1, N] \times \mathcal{I}[1, N]$,

$$\begin{bmatrix} A_i^\top P_j A_i - \alpha_{i,j} P_i & A_i^\top P_j B_i \\ * & B_i^\top P_j B_i - \frac{1-\alpha_{i,j}}{d^2} I \end{bmatrix} \preceq 0 \tag{5}$$

then system (1) is GUAS and the reachable set $\mathcal{R}_x$ can be over approximated by $\tilde{\mathcal{R}}_x \triangleq \bigcup_{i\in\mathcal{I}[1,N]} \mathcal{E}(P_i)$.

Remark 1: In [24], Lemma 1 has $V_i(x(k)) \leq 1$, $\forall i \in \mathcal{I}[1, N]$, and the reachable set $\mathcal{R}_x$ in Lemma 2 is bounded by the intersection of a set of ellipsoids $\bigcap_{i\in\mathcal{I}[1,N]} \mathcal{E}(P_i)$. We correct this slight error as that $\exists i \in \mathcal{I}[1, N]$ such that $V_i(x(k)) \leq 1$ and the over approximate set $\tilde{\mathcal{R}}_x$ should be the union of a set of ellipsoids $\bigcup_{i\in\mathcal{I}[1,N]} \mathcal{E}(P_i)$, since $\sigma(k)$ is an arbitrary switching means $\sigma(k)$ could be any possible $i \in \mathcal{I}[1, N]$ and $\tilde{\mathcal{R}}_x$ needs to include all possible trajectories for any $i \in \mathcal{I}[1, N]$.

On the basis of above lemma, the over approximate reachable set $\tilde{\mathcal{R}}_x$ can be characterized by a set of ellipsoids, and optimization problems can be formulated to obtain $\tilde{\mathcal{R}}_x$ as small as possible in [24]. In this paper, our main aim is to further improve this Lyapunov function based approach to develop less conservative result for reachable set estimation of switched system (1), namely to develop an approach to better over approximate the reachable set $\mathcal{R}_x$.

III. Main Results

In this section, the reachable set estimation problem will be studied based on a novel conception named M-step sequence, an LMI based approach will be proposed to obtain a set of bounding ellipsoids. Moreover, the globally uniformly asymptotical stability of discrete-time switched linear system is discussed in the framework of M-step sequence.

A. M-Step Sequence

In this paper, the main aim is to further reduce the conservatism in Lyapunov function based approach for reachable set estimation of discrete-time switched system over switched Lyapunov function methods. First, we introduce the conception of the M-step sequence, which plays a fundamental role in this paper. The M-step sequence is defined as follows.

Definition 1: For a switched system consisting N subsystem, and given a time window with M-step length, an M-step sequence is a combination of subsystems in M steps. There are N^M combinations of subsystems in M steps, and these N^M combinations are indexed by $\mathcal{I}[1, N^M]$. For the ith sequence of combination in $\mathcal{I}[1, N^M]$, it is expressed by

$$\mathcal{S}_i^M \triangleq \{i_1, i_2, \ldots, i_M\},\ i_1, \ldots, i_M \in \mathcal{I}[1, N],\ i \in \mathcal{I}[1, N^M]$$

The M-step sequence is able to characterize all possible activation orders for switched system during the M steps. Given a switching signal $\sigma(k)$ in $[0, \infty)$, we denote the nth M-step activation sequence is

$$\Sigma_n \triangleq \{\sigma(nM), \sigma(nM + 1), \ldots, \sigma((n + 1)M - 1)\} \tag{6}$$

where $n = 0, 1, \ldots$.

The following properties can be easily observed.

Proposition 1: Given any switching signal $\sigma(k)$ defined over interval $[0,\infty)$, we have

1) $\bigcup_{n=0}^{\infty}\Sigma_n=\{\sigma(0),\sigma(1),\sigma(2),\ldots\}$.
2) For any $n=0,1,\ldots$, there exists an $i\in\mathcal{I}[1,N^M]$ such that $\Sigma_n=\mathcal{S}_i^M$.

Remark 2: The first property implies the activation order of any switching signal $\sigma(k)$ can be expressed by M-step activation sequence Σ_n as $n\to\infty$. The second property means any M-step activation sequence Σ_n can be found in $\mathcal{S}_i^M$, $i\in\mathcal{I}[1,N^M]$. These two properties suffice to show that the M-step sequence $\mathcal{S}_i^M$, $i\in\mathcal{I}[1,N^M]$, is capable to describe the behaviors of switching signal $\sigma(k)$ in $[0,\infty)$.

Based on the introduced notion of M-step sequence, we introduce a class of M-step clock-dependent switched Lyapunov functions $V_i:[nM+1,(n+1)M]\times\mathbb{R}^n\to\mathbb{R}_+$, $n=0,1,\ldots$, $i\in\mathcal{I}[1,N^M]$, associated to M-step sequence, which are a family of non-negative functions satisfying

$$\beta_1(|x|)<V_i(k,x)<\beta_2(|x|) \tag{7}$$

where $\beta_1,\beta_2\in\mathcal{K}_\infty$.

In the framework of the M-step clock-dependent switched Lyapunov function, the following result can be obtained as an improvement of Lemma 1.

Theorem 1: Consider system (1) with input (2). If there exist a set of a family of non-negative functions $V_i:[nM+1,(n+1)M]\times\mathbb{R}^n\to\mathbb{R}_+$, $n=0,1,\ldots$, $i\in\mathcal{I}[1,N^M]$ satisfying (7), and exist scalars $0<\alpha_i,\alpha_{i,j}<1$ such that $\forall(i,j)\in\mathcal{I}[1,N^M]\times\mathcal{I}[1,N^M]$,

$$\Omega_i(k)\le 0,\ \forall k=nM+1,\ldots,(n+1)M \tag{8}$$

$$\Theta_{i,j}\le 0 \tag{9}$$

where $\Omega_i(k)=V_i(k+1,x(k+1))-\alpha_iV_i(k,x(k))-\frac{1-\alpha_i}{d^2}\omega^\top(k)\omega(k)$, $\Theta_{i,j}=V_j(nM+1,x(nM+1))-\alpha_{i,j}V_i(nM,x(nM))-\frac{1-\alpha_{i,j}}{d^2}\omega^\top(nM)\omega(nM)$, $n=1,2,\ldots$. Then system (1) is uniformly stable and we have $\exists i\in\mathcal{I}[1,N^M]$ such that $V_i(x(k))\le 1$ for all x_0 satisfying $V_i(0,x_0)\le 1$, $\forall i\in\mathcal{I}[1,N^M]$.

Proof: First, we consider $\omega(k)=0$ for stability. By (8), it ensures that

$$V_i(k+1,x(k+1))-\alpha_iV_i(k,x(k))\le 0 \tag{10}$$

holds for $k=nM+1,nM+1,\ldots,(n+1)M$.

Then, by (9), one has

$$V_j(nM+1,x(nM+1))-\alpha_{i,j}V_i(nM,x(nM))\le 0 \tag{11}$$

Define a new function $\tilde\sigma:\mathbb{N}\to\mathcal{I}[1,N^M]$ indicating the active M-step sequence, and choose a Lyapunov function candidate as $V_{\tilde\sigma(k)}(k,x(k))$. According to Proposition 1, and together with (10) and (11) with the fact $0<\alpha_i,\alpha_{i,j}<1$, it leads to

$$V_{\tilde\sigma(k+1)}(k+1,x(k+1))-V_{\tilde\sigma(k)}(k,x(k))<0 \tag{12}$$

Combined with (7), the stability can be established by standard Lyapunov theorem.

Furthermore, in presence of input $\omega(k)$, (8) yields that

$$\begin{aligned}V_i(k+1,x(k+1))-\alpha_iV_i(k,x(k))&\le\frac{1-\alpha_i}{d^2}\omega^\top(k)\omega(k)\\&\le 1-\alpha_i\end{aligned} \tag{13}$$

which implies $V_i(k+1,x(k+1))-1\le\alpha_i(V_i(k,x(k))-1)$.

Similarly, (9) can lead to

$$V_j(nM+1,x(nM+1))-\alpha_{i,j}V_i(nM,x(nM))\le 1-\alpha_i \tag{14}$$

holds for $n=1,2,\ldots$, which implies

$$V_j(nM+1,x(nM+1))-1\le\alpha_{i,j}(V_i(nM,x(nM))-1) \tag{15}$$

For any $k\in\mathbb{N}$, we have

$$\begin{aligned}&V_{\tilde\sigma(k)}(k)-1\\\le&\alpha_{\tilde\sigma(k-1)}(V_{\tilde\sigma(k-1)}(k-1)-1)\\\le&\alpha_{\tilde\sigma(k-1)}\cdots\alpha_{\tilde\sigma(nM+1)}(V_{\tilde\sigma(nM)}(nM+1)-1)\\\le&\alpha_{\tilde\sigma(k-1)}\cdots\alpha_{\tilde\sigma(nM+1)}\alpha_{\tilde\sigma(nM+1),\tilde\sigma(nM)}(V_{\tilde\sigma(nM)}(nM)-1)\\\le&\alpha_{\tilde\sigma(k-1)}\cdots\alpha_{\tilde\sigma(nM)}\alpha_{\tilde\sigma(nM),\tilde\sigma(nM-1)}\cdots\alpha_{\tilde\sigma(0)}(V_{\tilde\sigma(0)}(0)-1)\end{aligned}$$

Due to $0<\alpha_i,\alpha_{i,j}<1$ and $V_i(0,x_0)\le 1$, $\forall i\in\mathcal{I}[1,N^M]$, it ensures $V_{\tilde\sigma(k)}(k)-1\le 0$, $\forall k\in\mathbb{N}$. Because $\tilde\sigma(k)$ is an arbitrary signal selecting value in $\mathcal{I}[1,N^M]$, it implies $\exists i\in\mathcal{I}[1,N^M]\Rightarrow V_i(k,x)\le 1$. ■

Remark 3: If we particularly let $M=1$, Condition (8) is reduced to

$$V_i(n+1,x(n+1))-\alpha_iV_i(n,x(n))-\frac{1-\alpha_i}{d^2}\omega^\top(n)\omega(n)\le 0 \tag{16}$$

and (9) can be rewritten to

$$V_j(n+1,x(n+1))-\alpha_{i,j}V_i(n,x(n))-\frac{1-\alpha_{i,j}}{d^2}\omega^\top(n)\omega(n)\le 0 \tag{17}$$

It is noted that (16) can be absorbed in (17) by just letting $\alpha_{i,i}=\alpha_i$. It can be seen that (17) is exactly the condition (4) in Lemma 1. Therefore, Theorem 1 covers previous result stated by Lemma 1, namely Lemma 1 is a particular case which can be recovered by Theorem 1 with $M=1$.

B. Reachable Set Estimation

In this subsection, the reachable set estimation for discrete-time switched linear system will be investigated. Based on Theorem 1, the following result can be obtained.

Theorem 2: Consider system (1) with input (2). If there exist matrices $P_{i,m}\succ 0$, $m\in\mathcal{I}[1,M]$, $i\in\mathcal{I}[1,N^M]$ and scalars $0<\alpha_i<1$, $0<\alpha_{i,j}<1$ such that $\forall(i,j)\in\mathcal{I}[1,N^M]\times\mathcal{I}[1,N^M]$,

$$\begin{bmatrix}A_{i_m}^\top P_{i,m+1}A_{i_m}-\alpha_iP_{i,m} & A_{i_m}^\top P_{i,m}B_{i_m}\\ * & B_{i_m}^\top P_{i,m}B_{i_m}-\frac{1-\alpha_i}{d^2}I\end{bmatrix}\preceq 0,\quad m=1,2,\ldots,M-1 \tag{18}$$

$$\begin{bmatrix}A_{i_M}^\top P_{j,1}A_{i_M}-\alpha_{i,j}P_{i,M} & A_{i_M}^\top P_{j,1}B_{i_M}\\ * & B_{i_M}^\top P_{j,1}B_{i_M}-\frac{1-\alpha_{i,j}}{d^2}I\end{bmatrix}\preceq 0 \tag{19}$$

then system (1) is uniformly stable and the reachable set $\mathcal{R}_x$ can be over approximated by

$$\tilde{\mathcal{R}}_x \triangleq \bigcup_{m\in\mathcal{I}[1,M],i\in\mathcal{I}[1,N^M]} \mathcal{E}(P_{i,m}) \tag{20}$$

Proof: Choosing an M-step clock-dependent switched Lyapunov function in the following quadratic form

$$V_{\tilde{\sigma}(k)}(k,x(k)) = x^\top(k)P_{\tilde{\sigma}(k),k-nM+1}x(k),\ n=0,1,\dots \tag{21}$$

where $\tilde{\sigma}(k)$ indicating the active M-step sequence defined same as in (12).

Suppose $\tilde{\sigma}(k) = i$, $k \in [nM+1,(n+1)M]$ and denote $\Omega_i(k) = V_i(k+1,x(k+1)) - \alpha_i V_i(k,x(k)) - \frac{1-\alpha_i}{d^2}\omega^\top(k)\omega(k)$, and thus the M-step sequence $\mathcal{S}_i^M = \{i_1,i_2,\dots,i_M\}$. Along the system evolution, we can obtain

$$\Omega_i(k) = \xi^\top(k)\Xi_{i,m}\xi(k)$$

where $m \in \mathcal{I}[1,M]$, $\xi(k) = [x^\top(k),\omega^\top(k)]^\top$ and $\Xi_{i,m} = \begin{bmatrix} A_{i_m}^\top P_{i,m+1}A_{i_m} - \alpha_i P_{i,m} & A_{i_m}^\top P_{i_m}B_{i_m} \\ * & B_{i_m}^\top P_{i,m}B_{i_m} - \frac{1-\alpha_i}{d^2}I \end{bmatrix}$.

Moreover, assume $\tilde{\sigma}(nM) = i$ and $\tilde{\sigma}(nM+1) = j$, and let $\Theta_{i,j} = V_j(nM+1,x(nM+1)) - \alpha_{i,j}V_i(nM,x(nM)) - \frac{1-\alpha_{i,j}}{d^2}\omega^\top(nM)\omega(nM)$, the following derivation can be obtained for the transition from instant nM to $nM+1$.

$$\Theta_{i,j} = \xi^\top(nM)\Pi_{i,j}\xi(nM)$$

where

$$\Pi_{i,j} = \begin{bmatrix} A_{i_M}^\top P_{j,1}A_{i_M} - \alpha_{i,j}P_{i_M} & A_{i_M}^\top P_{j,1}B_{i_M} \\ * & B_{i_M}^\top P_{j,1}B_{i_M} - \frac{1-\alpha_{i,j}}{d^2}I \end{bmatrix}$$

By (18) and (19), it can be ensured that $\Omega_i(k) \le 0$, $\forall k = nM+1,\dots,(n+1)M$, $\forall n = 1,2,\dots$ and $\Theta_{i,j} \le 0$.

According to Theorem 1, for the case of $x_0 = 0$, we have $\exists i \in \mathcal{I}[1,N^M]$ such that $V_i(x(k)) \le 1$. Therefore, the state $x(k)$ satisfies $x \in \{x \mid x^\top P_{i,m}x \le 1, m \in \mathcal{I}[1,M], i \in \mathcal{I}[1,N^M]\} = \bigcup_{m\in\mathcal{I}[1,M],i\in\mathcal{I}[1,N^M]}\mathcal{E}(P_{i,m})$, which is exactly the set (20). The proof is complete. ■

Remark 4: Theorem 2 can be viewed as an improved version for Lemma 2, if we enforce $M = 1$ in Theorem 2, $P_{i,m}$, $m \in \mathcal{I}[1,M]$, $i \in \mathcal{I}[1,N^M]$, becomes P_i, $i \in \mathcal{I}[1,N]$. Inequalities (18) and (19) can be rewritten to

$$\begin{bmatrix} A_i^\top P_j A_i - \alpha_{i,j}P_i & A_i^\top P_j B_i \\ * & B_i^\top P_j B_i - \frac{1-\alpha_{i,j}}{d^2}I \end{bmatrix} \preceq 0$$

which is (5) in Lemma 2.

Remark 5: The set $\tilde{\mathcal{R}}_x$ is usually expected to be as small as possible to achieve a precise estimate of reachable set $\mathcal{R}_y$. In [24], several methods have been proposed to minimize the bounding ellipsoids, which can be also employed in our paper. In order to make a clear comparison with [24], we consider the method associated to the following constraint

$$P_{i,m} \succeq \epsilon I,\ \epsilon > 0,\ \forall m \in \mathcal{I}[1,M],\ \forall i \in \mathcal{I}[1,N^M] \tag{22}$$

which implies that $\epsilon x^\top(k)x(k) \le x^\top(t)P_{i,m}x(k) \le 1$, namely $x(t) \in \tilde{\mathcal{R}}_x \triangleq \bigcup_{m\in\mathcal{I}[1,M],i\in\mathcal{I}[1,N^M]}\mathcal{E}(P_{i,m}) \subseteq \mathcal{B}(0,1/\sqrt{\epsilon})$, $\forall k \in \mathbb{R}_{\ge 0}$, so we have to maximize ϵ to obtain a smallest ball $\mathcal{B}(0,1/\sqrt{\epsilon})$ as

$$\begin{aligned} &\max\ \epsilon \\ &\text{s.t. } (18),\ (19) \text{ and } (22) \end{aligned} \tag{23}$$

Moreover, due to the existence of tuning parameters α_i and $\alpha_{i,j}$, the result in Theorem 2 and corresponding optimization problem (23) are not standard LMI problems, they are bilinear matrix inequality (BMI) problems and known to be NP-hard. Fortunately, several algorithms are available to solve BMI problems such as the iterative linear matrix inequality (ILMI) approach in [26], [27], or using numerical optimization algorithms, such as program `fminsearch` [20] or genetic algorithm (GA) [24] in the optimization toolbox of Matlab.

Remark 6: Although $M > 1$ will reduce the conservativeness, the price to pay is the increase of computational complexity. The number of LMIs and involved decision variables grows as M is increased. The computation complexities are listed in Table I.

TABLE I
COMPUTATIONAL COMPLEXITY OF THEOREM 2

	Number of variables	Size of LMIs
Theorem 2	$\frac{n(n+1)MN^M}{2}$	$n(N^{2M}+MN^M)$

C. Some Discussions for Stability Analysis

It should be noted that the stability analysis result of switched system (1) with input $\omega(k) = 0$ is actually included in the previous reachable set estimation solution. As what has been shown in previous section, our reachable set estimation yields less conservativeness than that in [24] which is essentially based on switched Lyapunov function approach in [25] . In fact, by introducing the concept of M-step sequence, a less conservative stability analysis result can be obtained as well in contrast to the well known stability criterion proposed in [25] on basis of switched Lyapunov function approach.

The following corollary can be viewed as an improvement for the classical switched Lyapunov function approach in stability analysis.

Corollary 1: Consider switched system (1) with $\omega(k) = 0$, if there exist MN^M symmetric matrices $P_{i,m} \succ 0$, $m \in \mathcal{I}[0,M]$, $i \in \mathcal{I}[1,N^M]$ such that the following inequalities hold for $\forall i,j \in \mathcal{I}[1,N^M]$, $\forall m \in \mathcal{I}[1,M]$,

$$A_{i_{m+1}}^\top P_{i,m+1}A_{i_m} - P_{i,m} \prec 0,\ m = 1,2,\dots,M-1 \tag{24}$$

$$A_{i_M}^\top P_{j,1}A_{i_M} - P_{i,M} \prec 0 \tag{25}$$

then switched system (1) is globally uniformly asymptotically stable.

Proof: The proof can be obtained by the guidelines in Theorems 1 and 2, which is omitted here. ■

Remark 7: Corollary 1 can be viewed as an improved result over switched Lyapunov function approach for switched system (1). By letting $M = 1$, conditions (24) and (25) can be rewritten to

$$A_i^\top P_j A_i - P_i \prec 0,\ i, j \in \mathcal{I}[1, N] \tag{26}$$

where $P_i \succ 0$, $i \in \mathcal{I}[1, N]$. This result is exactly the Theorem 2 in [25], which means that the switched Lyapunov function approach is a special case of Corollary 1 as $M = 1$. Corollary 1 with $M \geq 2$ is able to yield less conservativeness in stability analysis, which can be shown by a numerical example later.

IV. Example

Example 1: Consider a switched system with two modes with the following system matrices

$$A_1 = \begin{bmatrix} 0 & 0.7 \\ -0.2 & -0.6 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.2 \\ -0.4 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -0.6 & 0.4 \\ -0.7 & 0.2 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.6 \\ 0.4 \end{bmatrix}$$

The disturbance $\omega(k)$ satisfies $\omega(k) \in \mathcal{W} \triangleq \{\omega \in \mathbb{R}^{n_\omega} \mid \omega^\top \omega \leq 1\}$. In order to compare our approach with that in [24], we first use Lemma 2 to obtain the reachable set estimation by maximizing ϵ in optimization (23). The GA is used to search for optimized α_i, $i \in \mathcal{I}[1, 2]$. The population is set to be 50. After 100 generations, the optimal $\epsilon = 0.04057$, which is shown in Fig. 1.

On the other hand, with same population and generation, Theorem 2 with $M = 2$ reaches a larger ϵ as $\epsilon = 0.05618$, which obviously is a less conservative result. The update of ϵ at each generation is illustrated in Fig. 1, which has a slower convergent rate but a better optimized result. The slower convergent rate is basically because more variables $\alpha_{i,j}$, $i, j \in \mathcal{I}[1, 4]$, are introduced in the optimization problem. The union of bounding ellipsoids are depicted in Fig. 2 by solid blue lines. For the purpose of showing the advantage of our approach, we present Fig. 3 to clearly compare Theorem 2 and Lemma 2, in which the estimation by Theorem 2 is more precise than by Lemma 2. In Figs. 2 and 3, the state trajectories are generated with arbitrary switching signal and disturbance $\omega(k)$ uniformly distributed over $[-1, 1]$.

Example 2: In this example, we will show the less conservativeness of M-step method in the stability point of view. Let us consider the system (1) with matrices $A_i = e^{B_i T}$, where

$$B_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -4 \end{bmatrix} \tag{27}$$

Letting $T = 0.1$, and using switched Lyapunov function approach in [25] (also viewed as $M = 1$ in our M-step sequence approach), it can be found that the LMI problem is not feasible, so that the globally uniformly asymptotically stability cannot be determined by the approach in [25]. Moreover, by applying the method in [28], the minimum admissible dwell time is computed as 2, which also indicates that the globally uniformly asymptotically stability of switched system (1) cannot be ascertained for the case

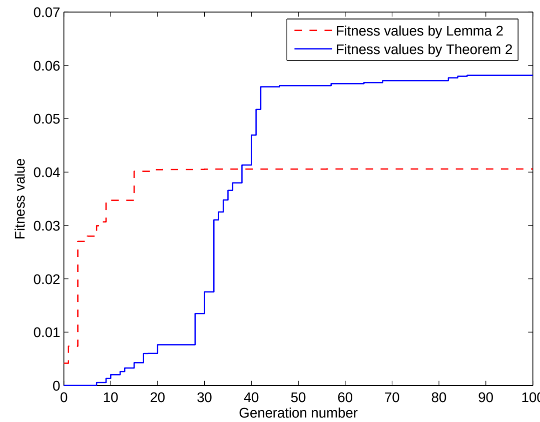

Fig. 1. Fitness function value along with generations.

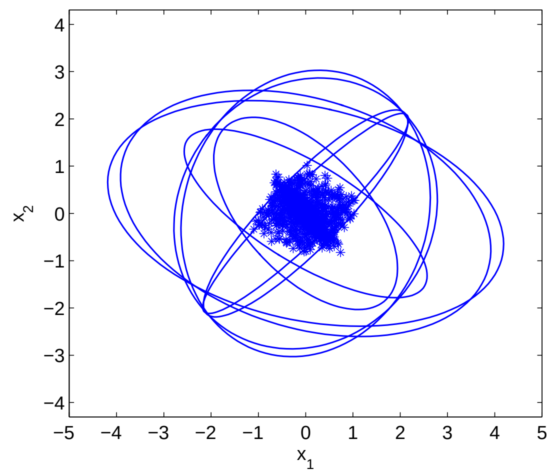

Fig. 2. Bounding ellipsoids by Theorem 2.

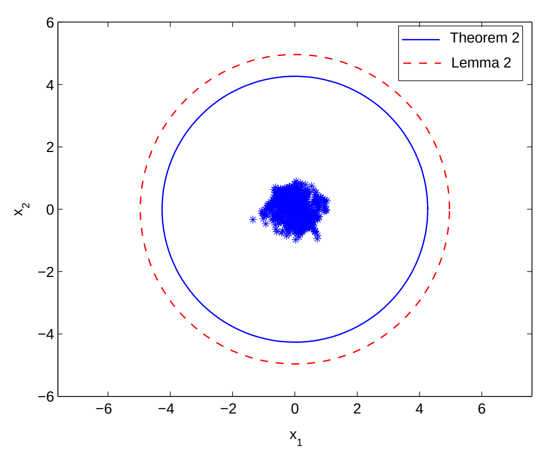

Fig. 3. Bounding circles by Lemma 2 and Theorem 2.

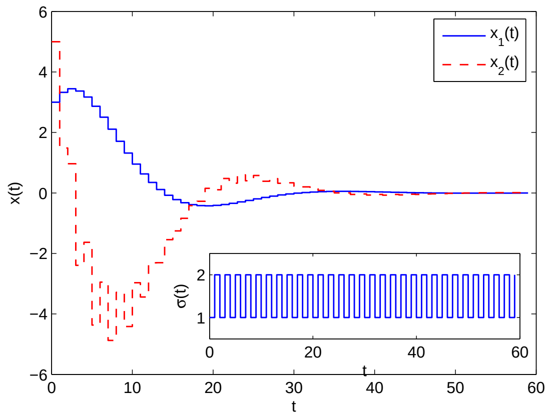

Fig. 4. State response under switching occurring at each time instant.

of arbitrary switching, for which the minimum dwell time should be 1.

However, if we increase M by just letting $M = 2$ in the M-step sequence method proposed in this paper, the feasibility of the corresponding LMI problems can be established, which is sufficient to guarantee that the system is globally uniformly asymptotically stable under arbitrary switching. The convergent state evolution is shown by the following simulation result in Fig. 4, where the extreme switching behavior, i.e., the switching occurs at each time instant, is adopted, and the initial state is assumed to be $x_0 = [3 \ \ 5]^\top$.

V. Conclusions

The reachable set estimation problem for discrete-time switched system has been investigated in this paper. A novel conception called M-step sequence is introduced to solve the reachable set estimation problem, it is shown that the proposed approach covers the previous result which is based on switched Lyapunov function, and thus has less conservativeness. In addition, some discussions are given for stability analysis for discrete-time switched system in the framework of M-step sequence. Finally, numerical examples are given to show the theoretical findings in this paper.

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