Event-Triggered Control for Continuous-Time Switched Linear Systems

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Abstract

The event-triggered control problem for switched linear system is addressed in this paper. The periodical sampling scheme and event-triggering condition are incorporated in the closed-loop. The feedback control updates its value only at sampling instants as long as event-triggering condition is satisfied as well. In addition, the switchings are only allowed to occur at sampling instants and meanwhile the switching condition is satisfied. Three equivalent sufficient conditions are proposed to ensure the asymptotic stability of switched systems. In particular, one condition has a promising feature of affineness in system matrices, and as a consequence, it is extended to robust sampling case and $\mathcal{L}_2$-gain analysis. Several examples are provided to illustrate our results.

Keywords: Asymptotic stability, event-triggered control, $\mathcal{L}_2$ gain, switched systems

1 Introduction

Switched systems have emerged as an important subclass of hybrid systems and represent a very active area of current research in the field of control systems [1–3]. A switched system is composed of a family of continuous or discrete-time subsystems, described by differential or difference equations, respectively, along with a switching rule governing the switching amongst the subsystems. The motivation for studying switched systems comes from the fact that switched system can be effectively used to model many practical systems that are inherently multi-model in the sense that several dynamic subsystem models are required to describe their behavior. For instance, the sampled data systems [4], networked control systems [5] and event-triggered systems [6] can be modeled as switched systems. Generally, the stability and stabilization problems are the main concerns in the field of switched systems. It has been proved that Lyapunov function techniques are effective to deal with stability and stabilization problems for switched systems, for example [7–9]. Combining multiple Lyapunov function (MLF), the dwell time and average dwell time properties of relatively slowly switched systems have been investigated in the corresponding switched systems [10–12]. For more details on the recent advances in the area, the readers are referred to the surveys [2], and the references cited therein.

On the other hand, the periodic and aperiodic control strategies are presented as the most prevailing control approaches on digital platforms. Typically, the control executes periodically in the closed-loop and the system can be analyzed by the well-developed sampled-data system theory. As a further

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improvement of traditional sampled-data system, the event-triggered control system is introduced, see for example the theory work [13–16], and numerous applications [17–20]. In the framework of event-triggered control, the control executions are generated by well-designed event-triggering condition. In comparison with sampled-data scheme, the event-triggered control which is a typical aperiodic one is capable of significantly reducing the number of control task executions, while retaining a satisfactory closed-loop performance. Though the event-triggered control can offers some clear advantages with respect to periodic control such as in handling energy, computation, and communication constraints but it also introduces some new theoretical and practical problems. The detailed advantages and challenges introduced by the event-triggered control can be found in the survey paper [21].

In this paper, we consider a class of periodic event-triggered control for switched linear systems. The periodic event-triggering condition allows the coexistence of periodic sampling scheme and event-triggering condition for the control executions. Moreover, this blending strategy also determines the occurrence of switching behaviors, in other words, the switching only occurs at sampling instants as long as the switching condition is satisfied. Three stability criteria are proposed for event-triggered switched system in this paper, and they are proved to be basically equivalent. The first one is derived by analyzing the evolution of state at sampling instant, however, it is not convenient to extend to further problems such as robust sampling and $\mathcal{L}_2$-gain analysis. Then, a sampling-dependent approach is proposed, which actually is not numerically tractable since it has infinitely many values to check. Thus, a discretized method to equivalently convert the sampling-dependent condition into a numerically tractable condition. Based on this numerically tractable condition, the extensions to robust sampling case and $\mathcal{L}_2$-gain analysis are made.

The remainder of this paper is organized as follows: The event-triggered switched system model is given in Section 2. The main result, three equivalent stability criteria are presented in Section 3. Extensions to robust sampling case and $\mathcal{L}_2$-gain analysis are studied in Section 4 and Section 5, respectively. Conclusions are given Section 6.

Notations: $\mathbb{N}$ represents the set of natural numbers, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^+$ is the set of nonnegative real numbers, and $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers, $\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices with real entries. The set $\mathbb{M}_c^n$ consists of all matrices $\Phi \in \mathbb{R}^{n\times n}$ with nonnegative off diagonal elements $\phi_{ji} \geq 0$, $i \neq j$, satisfying $\sum_{j=1}^{N} \phi_{ji} = 0$, which implies that $\phi_{ii} \leq 0$. The set $\mathbb{M}_d^n$ consists of all matrices $\Pi \in \mathbb{R}^{n\times n}$ with nonnegative elements $\pi_{ji} \geq 0$ satisfying the normalization constraints $\sum_{j=1}^{N} \pi_{ji} = 1$. $\|\cdot\|$ stands for Euclidean norm. The notation $A \succ 0$ means A is real symmetric and positive definite. $A \succ B$ means that $A - B \succ 0$. $A^\top$ denotes the transpose of A. In addition, in symmetric block matrices, we use $*$ as an ellipsis for the terms that are induced by symmetry and diag$\{\cdots\}$ stands for a block-diagonal matrix. I denotes the unit matrix and 0 stands for the zero elements in matrix with appropriate dimensions. We define $x(t_k^+) = \lim_{t\to t_k^+} x(t)$ and $x(t_k^-) = \lim_{t\to t_k^-} x(t)$. For a matrix function $F : [a, b] \to \mathbb{R}^{n\times n}$, its upper right Dini derivative is defined by $\mathcal{D}^+F(x) \triangleq \lim_{h\to 0^+} \sup \frac{F(x+h)-F(x)}{h}$. In the rest of this work, we will make extensive uses of the following matrix expressions:

$$
\begin{aligned}
\mathscr{C}(A, P) &= A^\top P^\top + PA \\
\mathscr{D}(A, P(t)) &= \mathscr{C}(A, P(t)) + \mathcal{D}^+P(t) \\
\mathscr{D}_1(A, P, Q, \delta) &= \mathscr{C}(A, P) + (P - Q)/\delta \\
\mathscr{D}_2(A, P, Q, \delta) &= \mathscr{C}(A, Q) + (P - Q)/\delta \\
\mathscr{E}(A, J, P, Q, t) &= e^{A^\top t} J^\top P J e^{At} - Q
\end{aligned}
$$

2 Event-Triggered Switched Control System

Consider the continuous-time switched linear system in the following form:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + E_{\sigma(t)}\omega(t) \tag{1}$$

$$y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}\omega(t) \tag{2}$$

where $x(t), x_0 \in \mathbb{R}^n$ are the state of the system and the initial condition, respectively. $u(t) \in \mathbb{R}^{n_u}$ is the input and $\omega(t) \in \mathbb{R}^{n_\omega}$ is the exogenous disturbance. $y(t) \in \mathbb{R}^{n_y}$ is the controlled output. The switching function $\sigma : \mathbb{R}^+ \to \mathcal{N} \triangleq \{1, 2, \ldots, N\}$ defines the switching actions, where N is the number of subsystems.

In this paper, we consider a periodic event-triggered control strategy for switched system (1)–(2) for the sake of taking advantages of both periodic sampled-data and event-triggered control, which means the system state $x(t)$ is only measured at the periodic sampling times for generating the control input, computing the switching function output and verifying the event-triggering condition. In a periodic sampling implementation, the values of the system state are available for a time sequence $\mathcal{S} \triangleq \{t_k\}_{k\in\mathbb{N}}$, where t_0 is the initial time and t_k, $k \in \mathbb{N} \setminus \{0\}$, are the sampling times, which are periodic in the sense that $t_k = kT_s$, $k \in \mathbb{N}$, for some properly chosen sampling interval $T_s > 0$. With this sampling setting, the sampled switching signal is

$$\sigma(t) = \hat{\sigma}(t),\ t \in (t_k, t_{k+1}] \tag{3}$$

where $\hat{\sigma}(t)$, $t \in (t_k, t_{k+1}]$, is determined by

$$\hat{\sigma}(t) = \begin{cases} \sigma(t_k) & \sigma(t_k) \neq \hat{\sigma}(t_k) \\ \hat{\sigma}(t_k) & \sigma(t_k) = \hat{\sigma}(t_k) \end{cases} \tag{4}$$

The sampled switching signal (3)–(4) implies the switching decisions are only made at sampling instant t_k. The value of $\sigma(t)$ only changes at sampling instant t_k if $\sigma(t_k) \neq \hat{\sigma}(t_k)$, otherwise it holds its most recent value. It worth mentioning that since the switching function (3) only activates at each sampling time t_k, $k \in \mathbb{N}$, it can be interpreted that a dwell time constraint $t_{k+1} - t_k \geq T_s$, $\forall k \in \mathbb{N}$ is imposed on the switching signal. This dwell time constraint obviously prevents the switching actions from chattering phenomenon or Zeno phenomenon, since the switching frequency is restricted to have an upper bound equals to $1/T_s$. In [22], a modified min-switching law with dwell time constraint is proposed to avoid the chattering behavior owe to the dwell time constraint. However, it requires accessing the system state and monitoring the state-dependent switching rule continuously, which is not allowed in the sampled-data setting proposed in this paper, since the system state $x(t)$ is obtained only at sampling instants.

In addition, we also take the sampled-data feedback controller into account. In a conventional periodic sampled-data control scheme, the following mode-dependent state feedback controller is often considered

$$u(t) = K_{\sigma(t)}\hat{x}(t),\ t \in \mathbb{R}^+ \tag{5}$$

where K_i, $i \in \mathcal{N}$ are the already designed feedback gains for subsystems, and $\hat{x}(t)$, $t \in (t_k, t_{k+1}]$, is defined by

$$\hat{x}(t) = x(t_k),\ t \in (t_k, t_{k+1}] \tag{6}$$

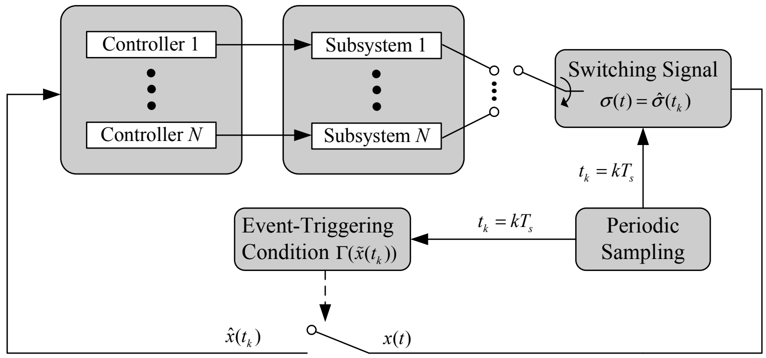

Figure 1: General scheme of periodic event-triggered switched control system

In order to obtain a complete model of system (1)–(2) with the periodic sampling setting (3) and (5), we let $\tilde{x}(t) = [x(t) \;\; \hat{x}(t)]^\top$ and obtain the following system

$$\dot{\tilde{x}}(t) = \tilde{A}_{\sigma(t)}\tilde{x}(t) + E_{\sigma(t)}\omega(t), \;\; t \in \mathbb{R}^+ \setminus \mathcal{S} \tag{7}$$

$$\tilde{x}(t_k^+) = J\tilde{x}(t_k^-), \;\; t_k \in \mathcal{S} \tag{8}$$

$$y(t) = \tilde{C}_{\sigma(t)}\tilde{x}(t) + \tilde{D}_{\sigma(t)}\omega(t) \tag{9}$$

where $\sigma(t)$ evolves according to (3) and

$$\tilde{A}_i = \begin{bmatrix} A_i & B_iK_i \\ 0 & 0 \end{bmatrix}, \; \tilde{E}_i = \begin{bmatrix} E_i \\ 0 \end{bmatrix}, \; J = \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix}, \; \tilde{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}, \; \tilde{D}_i = D_i$$

Further considering the event-triggered controller, the state measurements are transmitted over a communication network and the control values are updated only when certain event-triggering conditions are satisfied, the controller is given in the following form

$$u(t) = K_{\sigma(t)}\hat{x}(t), \;\; t \in \mathbb{R}^+ \tag{10}$$

where $\hat{x}(t)$ is a left-continuous signal, given for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}$, and modifies the (6) as

$$\hat{x}(t) = \begin{cases} x(t_k), & \Gamma(x(t_k), \hat{x}(t_k)) > 0 \\ \hat{x}(t_k), & \Gamma(x(t_k), \hat{x}(t_k)) \leq 0 \end{cases} \tag{11}$$

with an event-triggering function $\Gamma : \mathbb{R}^{2n} \to \mathbb{R}$. The value $\hat{x}(t_k)$ stands for the valid value for the controller at sampling time t_k and through the successive interval $[t_k, t_{k+1})$, which is determined by the event-triggering function Γ. If $\Gamma(x(t_k), \hat{x}(t_k)) \leq 0$, the state $\hat{x}(t_k)$ holds as its most recent value, and in the case of $\Gamma(x(t_k), \hat{x}(t_k)) > 0$, the state $x(t_k)$ is transmitted over the network to the controller and $\hat{x}(t_k)$ is updated accordingly. The general scheme of event-triggered switched control system with periodic sampling setting is illustrated in Figure 1.

In this paper, we focus on a class of quadratic event-triggering condition, that is, $\Gamma(x(t_k), \hat{x}(t_k))$ is in the following quadratic form

$$\Gamma(\tilde{x}(t_k)) = \tilde{x}^\top(t_k)Q\tilde{x}(t_k) \tag{12}$$

where $\tilde{x}(t_k) = [x^\top(t_k)\ \hat{x}^\top(t_k)]^\top$ and $Q \in \mathbb{R}^{2n\times 2n}$ is a symmetric matrix. Several event-triggering conditions can be written into the quadratic structure (12), for example the state-error based triggering condition $\Gamma(x(t_k), \hat{x}(t_k)) = \|\hat{x}(t_k) - x(t_k)\| - \Delta \|x(t_k)\|$, where $\Delta > 0$, can be expressed by (12) with

$$Q = \begin{bmatrix} (1-\Delta^2)I & -I \\ -I & I \end{bmatrix}$$

Other well-known event triggering conditions such as input-error based, Lyapunov function based conditions can be formalized by (12) as well, readers can refer to [6].

In summary, by modifying the periodic sampled-data system model (7)–(9), the event-triggered system model arrives at

$$\dot{\tilde{x}}(t) = \tilde{A}_{\sigma(t)}\tilde{x}(t) + \tilde{E}_{\sigma(t)}\omega(t),\ t \in \mathbb{R}^+ \setminus \mathcal{S} \tag{13}$$

$$\tilde{x}(t_k^+) = \begin{cases} J_1\tilde{x}(t_k^-), & \tilde{x}^\top(t_k^-)Q\tilde{x}(t_k^-) > 0 \\ J_2\tilde{x}(t_k^-), & \tilde{x}^\top(t_k^-)Q\tilde{x}(t_k^-) \le 0 \end{cases},\ t_k \in \mathcal{S} \tag{14}$$

$$y(t) = \tilde{C}_{\sigma(t)}\tilde{x}(t) + \tilde{D}_{\sigma(t)}\omega(t) \tag{15}$$

where J_1 is same as J in (8) and $J_2 = \mathrm{diag}\{I, I\}$.

By (13)–(15), one can see that the event-triggered switched control system can be expressed as a switched system with impulsive behaviors at switching instants. For the *passive* switching, that is the switching information is not available and the switching is supposed to possibly occur at every switching sampling instant, system (13)–(15) can be viewed to be under switching with a dwell time T_s. The results in [11, 23–25] for switched system with dwell time can be employed. However, if some *active* switching is considered, which means the switching rule is explicitly available to designed, the passive switching result could yields conservativeness, thus we should improve these results with the aid of the information of switching law. For the *active* switching considered in the remainder of paper, we adopt the well-known min-switching rule, which is described as below:

$$\sigma(t) = \arg\min_{i\in\mathcal{N}} \tilde{x}^\top(t)P_i\tilde{x}(t) \tag{16}$$

where $P_i \succ 0$, $i \in \mathcal{N}$, are matrices to be determined, see the results in [26, 27]. The corresponding sampled min-switching rule (16) is described as

$$\sigma(t) = \begin{cases} \arg\min_{i\in\mathcal{N}} \tilde{x}^\top(t_k^+)P_i\tilde{x}(t_k^+), & t_k \in \mathcal{S} \\ \sigma(t_k), & t \in (t_k, t_{k+1}) \end{cases} \tag{17}$$

The main aim of this paper is to provide analysis and design techniques for controller, sampling scheme, and event-triggering condition such that the system is stable with switching rule (17). In the following, the definition of globally asymptotic stability is presented.

Definition 1 *A function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{K}$ function if it is strictly increasing and $\gamma(0) = 0$, and also a function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{KL}$ function if for each fixed s the function $\beta(r, s)$ is a $\mathcal{K}$ function with respect to r, and for each fixed r the function $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to 0$.*

The definition of globally uniformly asymptotic stability (GUAS) for system (13)–(15) is given below.

Definition 2 *The equilibrium* $x = 0$ *of system (13)–(15) with* $\omega(t) = 0$ *is GUAS under the switching signal* $\sigma(t)$ *if, for initial condition* $\tilde{x}(t_0)$*, there exists a class* $\mathcal{KL}$ *function* β *such that the solution of the system satisfies* $\|\tilde{x}(t)\| \leq \beta(\|\tilde{x}(t_0)\|, t)$, $\forall t \in \mathbb{R}^+$.

In the presence of input $\omega(t)$, the $\mathcal{L}_2$-gain performance of system (13)–(15) is formulated in the following.

Definition 3 *For* $\gamma > 0$*, system (13)–(15) is said to be GUAS with an* $\mathcal{L}_2$*-gain performance, if the following is satisfied:*

(1) System (13)–(15) is GUAS when $\omega(t) = 0$*;*

(2) Under zero initial conditions, the following inequality holds for all nonzero $\omega \in \mathcal{L}_2[0, \infty)$,

$$\int_{t_0}^{\infty} \|y(t)\|^2 \,\mathrm{d}t \leq \gamma^2 \int_{t_0}^{\infty} \|\omega(t)\|^2 \,\mathrm{d}t \tag{18}$$

where γ *is called the* $\mathcal{L}_2$*-gain.*

Before ending this section, a useful lemma is introduced.

Lemma 1 *For a matrix* $A \in \mathbb{R}^{n\times n}$ *and a scalar* $T_s > 0$*, there always exist a sufficiently large* $M^* \in \mathbb{N} \setminus \{0\}$*, a sufficiently small* $\epsilon \in \mathbb{R}^+$ *and matrices* $P_m \in \mathbb{R}^{n\times n}$*,* $m = \{0, \ldots, M\}$*, such that*

$$P_m \succ 0,\ m \in \{0, \ldots, M\} \tag{19}$$

$$\mathscr{D}_1(A, P_{m+1}, P_m, \delta) \prec 0,\ m \in \{0, \ldots, M-1\} \tag{20}$$

$$\mathscr{D}_2(A, P_{m+1}, P_m, \delta) \prec 0,\ m \in \{0, \ldots, M-1\} \tag{21}$$

where $\delta = T_s/M$*, hold for any* $M > M^*$*, and* P_m*,* $m = \{0, \ldots, M\}$*, have the following form:*

$$P_m = e^{-A^\top \delta_m} P_0 e^{-A\delta_m} - \int_0^{\delta_m} e^{-A^\top(\delta_m - t)} Y(t) e^{-A(\delta_m - t)(t)} \mathrm{d}t,\ m \in \{0, \ldots, M\} \tag{22}$$

where $\delta_m = mT_s/M$*,* $m = \{0, \ldots, M\}$*, and* $0 \prec Y(t) \prec \epsilon I$*,* $t \in [0, T_s]$*.*

Proof. See Appendix. □

In this section, the closed-loop of event-triggered switched linear system is modeled as a switched system with state update at switching instant, along with mixed time-dependent and state-dependent switching rules. In the next section, the stability analysis will be studied as the main result in this paper.

3 Stability Analysis for Event-Triggered Switched System

Motivated by the techniques used in [23, 24, 28–30] for switched systems, and [31] for time-delayed systems, the main result for the stability of event-triggered switched control system (13)–(15) is presented by the following theorem.

Theorem 1 *Consider event-triggered switched control system (13)–(15) with* $\omega(t) = 0$*, the following three statements are equivalent:*

(a) There exist scalars $\mu_h > 0$, $h \in \{1,2\}$, a matrix $\Pi \in \mathbb{M}_d^N$ and symmetric matrices $P_i \succ 0$, $i \in \mathcal{N}$, such that

$$\Xi_{i,h} \prec 0, \ i \in \mathcal{N}, \ h \in \{1,2\} \tag{23}$$

where $\Xi_{i,h} = \mathscr{E}(\tilde{A}_i, J_h, \sum_{j=1}^{N} \pi_{ji} P_j, P_i + (-1)^h \mu_h \tilde{Q}_i, T_s)$, $\tilde{Q}_i = e^{\tilde{A}_i^\top T_s} Q e^{\tilde{A}_i T_s}$.

(b) There exist scalars $\mu_h > 0$, $h \in \{1,2\}$, a matrix $\Pi \in \mathbb{M}_d^N$ and a continuous symmetric matrix function $P_i(t) : [0, T_s] \to \mathbb{R}^{2n\times 2n}$, $i \in \mathcal{N}$, such that

$$P_i(t) \succ 0, \ t \in [0, T_s], \ i \in \mathcal{N} \tag{24}$$

$$\mathscr{D}(\tilde{A}_i, P_i(t)) \prec 0, \ i \in \mathcal{N} \tag{25}$$

$$\Omega_{i,h} \prec 0, \ i \in \mathcal{N}, \ h \in \{1,2\} \tag{26}$$

where $\Omega_{i,h} = J_h^\top \sum_{j=1}^{N} \pi_{ji} P_j(0) J_h - P_i(T_s) - (-1)^h \mu_h Q$.

(c) There exist scalars $M \in \mathbb{N} \setminus \{0\}$, $\mu_h > 0$, $h \in \{1,2\}$, a matrix $\Pi \in \mathbb{M}_d^N$ and symmetric matrices $P_{i,m} \in \mathbb{R}^{2n\times 2n}$, $m \in \{0, \ldots, M\}$, $i \in \mathcal{N}$, such that, for $i \in \mathcal{N}$,

$$P_{i,m} \succ 0, \ m \in \{0, \ldots, M\} \tag{27}$$

$$\mathscr{D}_1(\tilde{A}_i, P_{i,m+1}, P_{i,m}, \delta) \prec 0, \ m \in \{0, \ldots, M-1\} \tag{28}$$

$$\mathscr{D}_2(\tilde{A}_i, P_{i,m+1}, P_{i,m}, \delta) \prec 0, \ m \in \{0, \ldots, M-1\} \tag{29}$$

$$\Omega_{i,h} \prec 0, \ h \in \{1,2\} \tag{30}$$

where $\delta = T_s/M$ and $\Omega_{i,h} = J_h^\top \sum_{j=1}^{N} \pi_{ji} P_{j,0} J_h - P_{i,M} - (-1)^h \mu_h Q$.

when one of the above equivalent statements holds, then system (13)–(15) with $\omega(t) = 0$ is GUAS with switching signal (17) with P_i by statement (a), $P_i = P_i(0)$ by statement (b) and $P_i = P_{i,0}$ by statement (c), respectively.

Proof. The structure of the proof is as follows: First, we prove the equivalence by deriving (c) $\Rightarrow$ (b) $\Rightarrow$ (a) $\Rightarrow$ (c), then establish GUAS by (a) $\Rightarrow$ GUAS.

(c) $\Rightarrow$ (b): Dividing interval $\mathcal{I} \triangleq [0, T_s]$ can into $M \in \mathbb{N} \setminus \{0\}$ segments described as $\mathcal{I}_m \triangleq [\delta_m, \delta_{m+1})$, $m = 0, 1, \ldots, M-1$, which are of equal length $\delta = T_s/M$, and then $\delta_0 = 0$ and $\delta_m = m\delta = \frac{mT_s}{M}$. Based on the discretization of $\mathcal{I}$, the following time-scheduled matrices $P_i(t)$, $i \in \mathcal{N}$, are introduced

$$\begin{cases} P_i(t) = (1-\theta(t))P_{i,m} + \theta(t)P_{i,m+1} \\ \theta(t) = Mt/T_s - m \end{cases}, \ t \in \mathcal{I}_m \tag{31}$$

by which it can be seen that $0 \le \theta(t) \le 1$ and $P_i(t)$ defines a piecewise linear matrix function over $\mathcal{I}$.

By the definition of $P_i(t)$, $i \in \mathcal{N}$, as (31), we have $P_i(0) = P_{i,0}$ and $P_i(T_s) = P_{i,M}$, so (27) and (30) can make sure that (24) and (26) hold.

Then, one has

$$\mathcal{D}^+ P_i(t) = (P_{i,m+1} - P_{i,m})\mathcal{D}^+\theta(t), \ t \in \mathcal{I}_m \tag{32}$$

Due to $\theta(t) = M(t - \delta_m)/T_s$, we have $\mathcal{D}^+\theta(t) = M/T_s$. Hence $\mathcal{D}^+ P_i(t)$ becomes

$$\mathcal{D}^+ P_i(t) = M(P_{i,m+1} - P_{i,m})/T_s, \ t \in \mathcal{I}_m \tag{33}$$

Thus, (28) and (29) imply (25) holds.

(b) ⇒ (a): Pre- and post-multiplying (25) with $e^{\tilde{A}_i^\top t}$ and its transpose, and integrate it over $[0, T_s]$, it arrives

$$e^{\tilde{A}_i^\top T_s} P_i(T_s) e^{\tilde{A}_i T_s} - P_i(0) \prec 0,\ i \in \mathcal{N} \tag{34}$$

which implies $P_i(0) \succ e^{\tilde{A}_i^\top T_s} P_i(T_s) e^{\tilde{A}_i T_s}$, $i \in \mathcal{N}$. Furthermore, it equals to

$$P_i(T_s) \prec e^{-\tilde{A}_i^\top T_s} P_i(0) e^{-\tilde{A}_i T_s},\ i \in \mathcal{N} \tag{35}$$

Using (35) into (26), the following inequality holds for $i \in \mathcal{N}$ and $h \in \{1, 2\}$,

$$J_h^\top \sum\nolimits_{j=1}^{N} \pi_{ji} P_j(0) J_h - e^{-\tilde{A}_i^\top T_s} P_i(0) e^{-\tilde{A}_i T_s} - (-1)^h \mu_h Q \prec 0 \tag{36}$$

Letting $P_i = P_i(0) \succ 0$, $i \in \mathcal{N}$, (36) equals to

$$e^{\tilde{A}_i^\top T_s} J_h^\top \sum\nolimits_{j=1}^{N} \pi_{ji} P_j J_h e^{\tilde{A}_i T_s} - P_i - (-1)^h \mu_h \tilde{Q}_i \prec 0 \tag{37}$$

where $\tilde{Q}_i = e^{\tilde{A}_i^\top T_s} Q e^{\tilde{A}_i T_s}$. Thus, (23) can be established by letting $P_i = P_i(0) \succ 0$, $i \in \mathcal{N}$.

(a) ⇒ (c): Since (23) holds, it implies that the following inequality holds

$$J_h^\top \sum\nolimits_{j=1}^{N} \pi_{ji} P_j J_h - e^{-\tilde{A}_i^\top T_s} P_i e^{-\tilde{A}_i T_s} - (-1)^h \mu_h Q \prec 0 \tag{38}$$

which implies that there exists an $\epsilon^* > 0$ such that

$$J_h^\top \sum\nolimits_{j=1}^{N} \pi_{ji} P_j J_h - e^{-\tilde{A}_i^\top T_s} P_i e^{-\tilde{A}_i T_s} - (-1)^h \mu_h Q \prec -\epsilon^* I \tag{39}$$

Then, for any $\epsilon > 0$, we can let $P_{i,0} = \epsilon P_i / \epsilon^* \succ 0$, $i \in \mathcal{N}$ (This choice of $P_{i,0}$, $i \in \mathcal{N}$, maintains the same switching law generated by P_i, $i \in \mathcal{N}$.), and $\hat{\mu}_h = \epsilon \mu_h / \epsilon^* > 0$, $h \in \{1, 2\}$, such that

$$J_h^\top \sum\nolimits_{j=1}^{N} \pi_{ji} P_{j,0} J_h - e^{-\tilde{A}_i^\top T_s} P_{i,0} e^{-\tilde{A}_i T_s} - (-1)^h \hat{\mu}_h Q \prec -\epsilon I \tag{40}$$

Using Lemma 1, there always exists a sufficiently large M^* such that (27), (28), (29) always hold with $P_{i,m}$, $m \in \{0, \ldots, M\}$, $M > M^*$, $i \in \mathcal{N}$, in the form of

$$P_{i,m} = e^{-\tilde{A}_i^\top \delta_m} P_{i,0} e^{-\tilde{A}_i \delta_m} - Z_{i,m},\ m \in \{0, \ldots, M\} \tag{41}$$

where

$$Z_{i,m} = \int_0^{\delta_m} e^{-\tilde{A}_i^\top (\delta_m - t)} Y_i(t) e^{-\tilde{A}_i (\delta_m - t)(t)} \mathrm{d}t$$

with $\delta_m = m T_s / M$, $m = \{0, \ldots, M\}$, and continuous matrix functions $Y_i(t) \succ 0$, $i \in \mathcal{N}$.

Thus, it yields

$$P_{i,M} = e^{-\tilde{A}_i^\top T_s} P_{i,0} e^{-\tilde{A}_i T_s} - Z_{i,M} \tag{42}$$

Substituting $e^{-\tilde{A}_i^\top T_s} P_{i,0} e^{-\tilde{A}_i T_s} = P_{i,M} + Z_{i,M}$ into (40), we have

$$J_h^\top \sum\nolimits_{j=1}^{N} \pi_{ji} P_{j,0} J_h - P_{i,M} - (-1)^h \hat{\mu}_h Q \prec -\epsilon I + Z_{i,M} \tag{43}$$

Since $\epsilon > 0$ can be arbitrarily chosen, we can choose a sufficiently large $\epsilon > 0$ such that

$$J_h^\top \sum\nolimits_{j=1}^{N} \pi_{ji} P_{j,0} J_h - P_{i,M} - (-1)^h \hat{\mu}_h Q \prec 0 \tag{44}$$

which implies that (30) holds.

(a) $\Rightarrow$ GUAS: First, we consider the system state $\tilde{x}(t_k^+)$ at sampling instants, we have

$$\tilde{x}(t_{k+1}^+) = \begin{cases} J_1 e^{\tilde{A}_{\sigma(t_k^+)} T_s} \tilde{x}(t_k^+), & \tilde{x}^\top(t_{k+1}^-) Q \tilde{x}(t_{k+1}^-) > 0 \\ J_2 e^{\tilde{A}_{\sigma(t_k^+)} T_s} \tilde{x}(t_k^+), & \tilde{x}^\top(t_{k+1}^-) Q \tilde{x}(t_{k+1}^-) \le 0 \end{cases} \tag{45}$$

Due to $\tilde{x}(t_{k+1}^-) = e^{\tilde{A}_{\sigma(t_k^+)} T_s} \tilde{x}(t_k^+)$, and letting $\tilde{Q}_i = e^{\tilde{A}_i^\top T_s} Q e^{\tilde{A}_i T_s}$, $k = t_k^+$, $\tilde{x}(k)$ evolves according to the following dynamics

$$\tilde{x}(k+1) = \begin{cases} J_1 e^{\tilde{A}_{\sigma(k)} T_s} \tilde{x}(k), & \tilde{x}^\top(k) \tilde{Q}_i \tilde{x}(k) > 0 \\ J_2 e^{\tilde{A}_{\sigma(k)} T_s} \tilde{x}(k), & \tilde{x}^\top(k) \tilde{Q}_i \tilde{x}(k) \le 0 \end{cases} \tag{46}$$

where $\sigma(k) = \arg\min_{i \in \mathcal{N}} \tilde{x}^\top(k) P_i \tilde{x}(k)$ and P_i, $i \in \mathcal{N}$, is same as in switching signal (17).

Construct Lyapunov function candidate as $V(\tilde{x}(k)) = \tilde{x}^\top(k) P_{\sigma(k)} \tilde{x}(k)$ and define $\Delta V(\tilde{x}(k)) = V(\tilde{x}(k+1)) - V(\tilde{x}(k))$, under the min-switching law (17), we have

$$\begin{aligned} \Delta V(\tilde{x}(k)) &= \min_{j \in \mathcal{N}} \tilde{x}^\top(k+1) P_j \tilde{x}(k+1) - \tilde{x}^\top(k) P_i \tilde{x}(k) \\ &\le \tilde{x}^\top(k+1) \left(\sum\nolimits_{j=1}^{N} \pi_{ji} P_j \right) \tilde{x}(k+1) - \tilde{x}^\top(k) P_i \tilde{x}(k) \end{aligned}$$

By (46), $\Delta V(\tilde{x}(k))$ arrives

$$\Delta V(\tilde{x}(k)) = \begin{cases} \tilde{x}^\top(k) \Gamma_{i,1} \tilde{x}(k), & \tilde{x}^\top(k) \tilde{Q}_i \tilde{x}(k) > 0 \\ \tilde{x}^\top(k) \Gamma_{i,2} \tilde{x}(k), & \tilde{x}^\top(k) \tilde{Q}_i \tilde{x}(k) \le 0 \end{cases} \tag{47}$$

where $\Gamma_{i,1} = \mathscr{E}(\tilde{A}_i, J_1, \sum_{j=1}^{N} \pi_{ji} P_{j,0}, P_i, T_s)$, $\Gamma_{i,2} = \mathscr{E}(\tilde{A}_i, J_2, \sum_{j=1}^{N} \pi_{ji} P_{j,0}, P_i, T_s)$. Since (23) holds, it implies there exists a sufficiently small $\epsilon > 0$ such that $\Xi_{i,h} < -\epsilon I$, $\forall i \in \mathcal{N}$, $h \in \{1, 2\}$ then using *S-Procedure*, it ensures that

$$\Delta V(\tilde{x}(k)) < -\epsilon \left\| \tilde{x}(k) \right\|^2, \ k \in \mathbb{N} \tag{48}$$

Letting $\lambda_{\min}$, $\lambda_{\max}$ be the minimal and maximal eigenvalues of P_i, $i \in \mathcal{N}$, respectively, it implies that $\lambda_{\min} \left\| \tilde{x}(k) \right\|^2 \le V(\tilde{x}(k)) \le \lambda_{\max} \left\| \tilde{x}(k) \right\|^2$. Thus, (48) implies that $V(\tilde{x}(k)) < (1 - \epsilon/\lambda_{\max})^k V(\tilde{x}(t_0))$, where $0 < 1 - \epsilon/\lambda_{\max} < 1$. Due to $k = t_k/T_s$, one has

$$V(\tilde{x}(t_k^+)) < e^{(t_k - t_0) \ln \frac{1 - \epsilon/\lambda_{\max}}{T_s}} V(\tilde{x}(t_0)), \ t_k \in \mathcal{S} \tag{49}$$

Furthermore, it arrives

$$\left\| \tilde{x}(t_k^+) \right\| < \sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}} e^{-\rho(t_k - t_0)} \left\| \tilde{x}(t_0) \right\|, \ t_k \in \mathcal{S} \tag{50}$$

where $\rho = -\ln(1 - \epsilon/\lambda_{\max})/2T_s > 0$.

Then, let us consider any $t \in (t_k, t_{k+1})$, the dynamics of mode i yields $\tilde{x}(t) = e^{\tilde{A}_i (t - t_k)} \tilde{x}(t_k^+)$, $t \in (t_k, t_{k+1})$. Using the following derivation

$$\left\| e^{\tilde{A}_i (t - t_k)} \right\| \le e^{\left\| \tilde{A}_i (t - t_k) \right\|} \le e^{\left\| \tilde{A}_i \right\| T_s}, \ t \in (t_k, t_{k+1}) \tag{51}$$

we have $\left\| \tilde{x}(t) \right\| \le c \left\| \tilde{x}(t_k) \right\|$, $t \in (t_k, t_{k+1})$, where $c = \max_{i \in \mathcal{N}} e^{\left\| \tilde{A}_i \right\| T_s}$. Thus, by (50), it can be obtained that $\left\| \tilde{x}(t) \right\| < C e^{-\rho(t - t_0)} \left\| \tilde{x}(t_0) \right\|$, where $C = c e^{\rho T_s} \sqrt{\lambda_{\max}/\lambda_{\min}} > 0$, and the GUAS can be established by the existence of $\mathcal{KL}$ function $\beta(\left\| \tilde{x}(t_0) \right\|, t) = C e^{-\rho(t - t_0)} \left\| \tilde{x}(t_0) \right\|$. □

Some observations are obtained for three conditions in Theorem 1:

1. If no event-triggering condition is considered and the state $x(t)$ updates at each sampling instant, event-triggered system (13)–(15) is reduced to (7)–(9), and as a result, (23) can be rewritten to

$$\mathscr{E}(\tilde{A}_i, J, \sum\nolimits_{j=1}^{N} \pi_{ji} P_j, P_i, T_s) \prec 0,\ i \in \mathcal{N} \tag{52}$$

   It can be found that (52) recovers the result in [32], which deals with the switched system with min-switching law (16) only acts at sampling instant t_k. Theorem 1 generalizes the sampled switching case to event-triggered switching case. Furthermore, if we consider the *passive* switching, which means switched system could switch to any subsystems at every switching instant t_k. That means, for any $j \neq i,\ i, j \in \mathcal{N}$, we have to let $\pi_{ji} = 1$ and $\pi_{pi} = 0$, $p \neq j$, so

$$\mathscr{E}(\tilde{A}_i, J, P_j, P_i, T_s) \prec 0,\ i, j \in \mathcal{N} \tag{53}$$

   which exactly recovers the result in [23].

   The basic idea of Condition (a) is to consider the evolution of system state at sampling instant t_k, and the asymptotic convergence of $\tilde{x}(t_k)$ guarantees the asymptotic stability of system (13)–(15). However, if one attempts to make some further extensions of Condition (a) such as robust sampling case and $\mathcal{L}_2$-gain performance analysis, the presence of exponential term $e^{\tilde{A}_i T_s}$ makes such extensions difficult.

2. Condition (b) basically is an extension of the result in [28], from dwell time switching to periodically event-triggered switching. Regardless of event-triggering condition, system (7)–(9) is a switched system with a periodic dwell time T_s, and if we deactivate the switching rule (17) to consider *passive* switching, it leads to $\pi_{ji} = 1$ and $\pi_{pi} = 0$, $p \neq j$, thus (26) is rewritten to

$$P_j(0) - P_i(T_s) \prec 0,\ j \neq i,\ i, j \in \mathcal{N} \tag{54}$$

   Together with (24), (25), the result in [28] is recovered.

   Still consider system (7)–(9) regardless of event-triggering condition, (26) becomes

$$\sum\nolimits_{j=1}^{N} \pi_{ji} P_j(0) - P_i(T_s) \prec 0 \tag{55}$$

   Then, let us consider the special case with sampling interval $T_s \to 0$. In this case, we have to let the continuous matrix function $P_i(t) = P_i$, $i \in \mathcal{N}$, then (25) implies $\mathscr{D}(\tilde{A}_i, P_i) = \mathscr{C}(\tilde{A}_i, P_i)$, and (26) arrives at

$$\sum\nolimits_{j=1}^{N} \pi_{ji} P_j - P_i \prec 0 \tag{56}$$

   From the fact of $\sum\nolimits_{j=1}^{N} \pi_{ji} P_j - P_i = \sum\nolimits_{j=1}^{N} \phi_{ji} P_j$, $\Phi \in \mathbb{M}_c^N$, $i, j \in \mathcal{N}$, (56) leads to

$$\sum\nolimits_{j=1}^{N} \phi_{ji} P_j \prec 0 \tag{57}$$

   Combining (25), (57), the following result can be established

$$\mathscr{C}(\tilde{A}_i, P_i) + \sum\nolimits_{j=1}^{N} \phi_{ji} P_j \prec 0,\ \Phi \in \mathbb{M}_c^N,\ i, j \in \mathcal{N} \tag{58}$$

   which exactly recovers result in [23] for min-switching rule. Therefore, Condition (b) is an extension to sampling case and further to event-triggered case. One point need to be noted this

Table 1: Computational complexities of Conditions (a), (b) and (c) with fixed $\Pi \in \mathbb{M}_d^N$

	Number of Variables	LMI Constraints
Condition (a)	$(4n^2N + 2nN)/2 + 2$	$6nN + 2$
Condition (b)	∞	∞
Condition (c)	$(4n^2 + 2n)(M + 1)/2 + 2$	$6nN(M + 1) + 2$

min-switching law may introduce Zeno behaviors, but if we let T_s be a positive constant in our periodic event-triggered rule, one advantage is the elimination of Zeno behavior in switching.

In comparison with Condition (a), Condition (b) does not have any exponential terms which facilitates its further extensions to solve other problems. However, it is not numerically testable to check the existence of such time-varying matrix functions $P_i(t)$, $i \in \mathcal{N}$.

3. Condition (c) is a discretized version Condition (b), and similar as what has been discuss for Condition (b), if we discard the event-triggering condition and (30) becomes

$$\sum\nolimits_{j=1}^{N} \pi_{ji} P_{j,0} - P_{i,M} \prec 0 \tag{59}$$

which recovers the result in [22]. Moreover, if we further deactivate the min-switching strategy, (30) can be reduced to

$$P_{j,0} - P_{i,M} \prec 0 \tag{60}$$

to recover the result in [24] for switched system under dwell time constraint.

With a particularly constructed $P_i(t)$, $i \in \mathcal{N}$, Condition (c) recasts the search for a continuous matrix function $P_i(t)$ as a finite number of matrices $P_{i,m}$, $m \in \{0, \ldots, M\}$, $i \in \mathcal{N}$, which is solvable for many current tools.

4. Though the three conditions are equivalent, the computation complexities are different. Condition (a) looks simpler and computationally much more efficient, see Table 1 for the comparison of computational complexities with a prescribed $\Pi \in \mathbb{M}_d^N$. Condition (b) is actually not numerically tractable by the present tools, so a special structure of $P_i(t)$, $i \in \mathcal{N}$, is employed in Condition (c), it turns the infinite number of decision variables in time-varying $P_i(t)$, $i \in \mathcal{N}$ into a finite number of matrices $P_{i,m}$, $m \in \{0, \ldots, M\}$, $i \in \mathcal{N}$. However, the equivalency of Condition (c) to Conditions (a) and (b) has to be established based on a sufficient large M, and the computation cost increases as M grows, see Table 1. Though more computation cost has to pay in Condition (c), the further extensions beyond stability become possible.

Example 1 *Consider a switched system with two modes*

$$\left[\begin{array}{c} A_1 \\ \hdashline B_1^\top \end{array}\right] = \left[\begin{array}{cc} 1 & 3 \\ 6 & -2 \\ \hdashline 1 & 0.5 \end{array}\right], \quad \left[\begin{array}{c} A_2 \\ \hdashline B_2^\top \end{array}\right] = \left[\begin{array}{cc} -1.3 & -1.6 \\ -3.3 & 0.3 \\ \hdashline 0.2 & 0.3 \end{array}\right]$$

The feedback gains are $K_1 = [-5.1744 \quad -5.1904]$ and $K_2 = [18.7593 \quad 16.3442]$, which ensure the $A_i + B_iK_i$, $i \in \{1, 2\}$, are Hurwitz stable. The event triggering condition is $\Gamma(x(t_k), \hat{x}(t_k)) = \|\hat{x}(t_k) - x(t_k)\| - \Delta \|x(t_k)\|$, where $\Delta > 0$. To search for $\Pi \in \mathbb{M}_d^N$, we define $\pi_{11} \in [0, 1]$ and $\pi_{12} \in [0, 1]$, then $\pi_{21} = 1 - \pi_{11}$ and $\pi_{22} = 1 - \pi_{12}$, respectively. The increments $d\pi_{11} = 0.1$ and

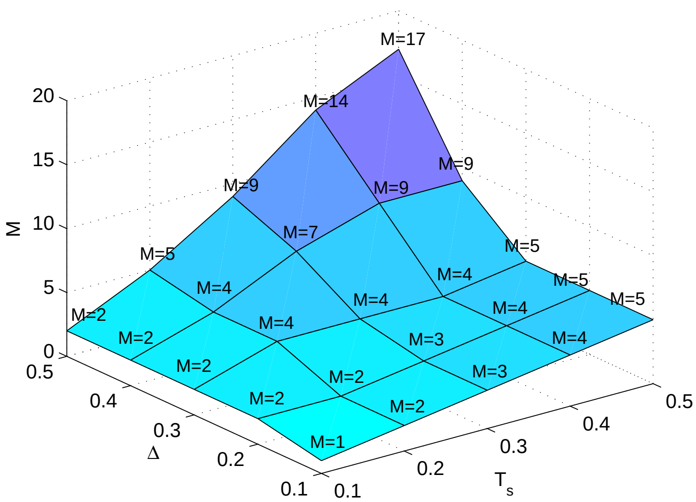

Figure 2: The least values of sufficiently large parameter M for Condition (c) to verify GUAS

Table 2: Computational time (second) of Condition (c) with a fixed $\Pi \in \mathbb{M}_d^N$

	$T_s = 0.1$	$T_s = 0.2$	$T_s = 0.3$	$T_s = 0.4$	$T_s = 0.5$
$\Delta = 0.1$	3.045	4.842	6.235	7.682	12.372
$\Delta = 0.2$	3.767	4.881	6.349	8.628	13.680
$\Delta = 0.3$	3.624	6.349	8.932	9.158	14.046
$\Delta = 0.4$	3.814	5.817	9.434	16.745	16.750
$\Delta = 0.5$	3.983	9.738	12.186	20.909	29.081

$d\pi_{12} = 0.1$ are taken to divide $[0, 1]$, and use the discretized points to turn the conditions in Conditions (a) and (c) into LMI feasibility problems.

First, we use Condition (a) to verify that the GUAS can be established with sampling times $T_s = \{0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5\}$ and state error $\Delta = \{0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5\}$. Then, to show the equivalence, we use Condition (c) to obtain same GUAS results, provided with sufficiently large parameters M. The results are shown in Figure 2.

Figure 2 shows the existence of sufficiently large M ensuring the equivalence of Conditions (a) and (c). However, the computational complexities of two theorems are different. The computational complexity of Condition (a) is fixed if the number of subsystems and system order are fixed, as Table 1 shows, but the computational complexity of Condition (c) increases as M grows. The computational time is given in Table 2. Larger Δ or T_s will lead to more computational time which is listed in Table 2 is because larger Δ or T_s needs larger M to establish the stability, as what Figure 1 shows. Taking the $T_s = 0.2$ for example, $\Delta = 0.2$ needs $M = 2$ and, on the other hand, $\Delta = 0.3$ needs $M = 4$. Larger M has more computational complexities as shown in Table 1. If the M are same, e.g. the case $T_s = 0.1$, $\Delta = 0.2$ and $\Delta = 0.3$ both need $M = 2$, so the computational times are similar.

Despite the equivalence of Conditions (a), (b) and (c), the main advantage of Condition (c) lies in its convenience of extending to solve further problems. In next sections, extensions will be made to robust sampling case and $\mathcal{L}_2$-gain performance analysis for event-triggered switched control system based on Condition (c).

4 Robust Sampling Scheme

In this section, the uncertainties in sampling interval will be considered. To further develop a robust switching rule (17), the sampling interval is generalized to $T_s \in [T_{\min}, T_{\max}]$. Similar as the generalization from periodic switching to aperiodic switching in [30], the generalization of Conditions (a) and (b) in Theorem 1 can be made simply by replace a fixed T_s by a variable $\tau \in [T_{\min}, T_{\max}]$ in these conditions. For instance, Condition (a) can be directly generalized as

$$\mathscr{E}(\tilde{A}_i, J_h, \sum\nolimits_{j=1}^{N} \pi_{ji} P_j, P_i + (-1)^h \mu_h e^{\tilde{A}_i^\top \tau} Q e^{\tilde{A}_i \tau}, \tau) \prec 0,\ i \in \mathcal{N},\ h \in \{1, 2\} \tag{61}$$

holds for all $\tau \in [T_{\min}, T_{\max}]$. However, it is difficult to check (61) for all $\tau \in [T_{\min}, T_{\max}]$ which has infinitely many number for checking in an interval $[T_{\min}, T_{\max}]$, due to the continuity argument and intricate dependence of (61) with $\tau \in [T_{\min}, T_{\max}]$. Thus, it is difficult to numerically verify the stability by (61) which actually requires infinite values for checking.

In order to establish a numerically tractable method for robust sampling interval $T_s \in [T_{\min}, T_{\max}]$, we resort to generalize Condition (c). Like the extension from dwell time to ranged dwell time in [30] for sampled-data systems, the following theorem can be developed for robust sampling interval in the framework of event-triggered control scheme.

Theorem 2 *Consider event-triggered switched control system (13)–(15) with $\omega(t) = 0$, if there exist scalars $M \in \mathbb{N} \setminus \{0\}$, $\mu_h > 0$, $h \in \{1, 2\}$, a matrix $\Pi \in \mathbb{M}_d^N$ and symmetric matrices $P_{i,m} \in \mathbb{R}^{2n \times 2n}$, $m \in \{0, \ldots, M\}$, $i \in \mathcal{N}$, such that, for $i \in \mathcal{N}$,*

$$P_{i,m} \succ 0,\ m \in \{0, \ldots, M\} \tag{62}$$

$$\mathscr{D}_1(\tilde{A}_i, P_{i,m+1}, P_{i,m}, \delta) \prec 0,\ m \in \{0, \ldots, M-1\} \tag{63}$$

$$\mathscr{D}_2(\tilde{A}_i, P_{i,m+1}, P_{i,m}, \delta) \prec 0,\ m \in \{0, \ldots, M-1\} \tag{64}$$

$$\Omega_{i,h,\hat{m}} \prec 0,\ \hat{m} \in \{\underline{M}, \ldots, M\},\ h \in \{1, 2\} \tag{65}$$

where $\delta = T_{\max}/M$, $\underline{M} = \mathrm{int}\{\frac{MT_{\min}}{T_{\max}}\}$ and $\Omega_{i,h,\hat{m}} = J_h^\top \sum_{j=1}^{N} \pi_{ji} P_{j,0} J_h - P_{i,\hat{m}} - (-1)^h \mu_h Q$, then system (13)–(15) with $\omega(t) = 0$ is GUAS under sampled switching rule (17) with $P_i = P_{i,0}$, $i \in \mathcal{N}$.

Proof. Since $\underline{M} = \mathrm{int}\{\frac{MT_{\min}}{T_{\max}}\}$, we have $\frac{\underline{M}T_{\max}}{M} \leq T_{\min}$ which implies that the interval $[T_{\min}, T_{\max}] \subseteq \bigcup_{\hat{m}=\underline{M},\ldots,M-1} \mathcal{I}_{\hat{m}}$.

Considering $P_i(t)$, $t \in [0, T_{\max}]$ defined by

$$\begin{cases} P_i(t) = (1 - \theta(t))P_{i,m} + \theta(t)P_{i,m+1} \\ \theta(t) = Mt/T_{\max} - m \end{cases},\ t \in \mathcal{I}_m \tag{66}$$

where $0 \leq \theta(t) \leq 1$. First by (62), we can obtain $P_i(t) \succ 0$, $t \in [T_{\min}, T_{\max}]$. Then, (63) and (64) have $\mathcal{D}(\tilde{A}_i, P_i(t)) \prec 0$, and for any $\tau \in [T_{\min}, T_{\max}]$, it is obtained

$$P_{i,0} = P_i(0) \succ e^{\tilde{A}_i^\top \tau} P_i(\tau) e^{\tilde{A}_i \tau},\ \tau \in [T_{\min}, T_{\max}] \tag{67}$$

by integrating $\mathcal{D}(\tilde{A}_i, P_i(t)) \prec 0$ over $[0,\tau]$. Then, (65) implies that

$$e^{\tilde{A}_i^\top \tau} J_h^\top \sum\nolimits_{j=1}^{N} \pi_{ji} P_{j,0} J_h e^{\tilde{A}_i^\top \tau} - e^{\tilde{A}_i^\top \tau} P_i(\tau) e^{\tilde{A}_i^\top \tau} - (-1)^h \mu_h \tilde{Q}(\tau) \prec 0 \tag{68}$$

holds for $\tau \in [T_{\min}, T_{\max}]$, where $\tilde{Q}_i(\tau) = e^{\tilde{A}_i^\top \tau} Q e^{\tilde{A}_i \tau}$. Using (67) into (68) and letting $P_i = P_{i,0}$, $i \in \mathcal{N}$, it reaches that

$$\mathscr{E}(\tilde{A}_i, J_h, \sum\nolimits_{j=1}^{N} \pi_{ji} P_j, P_i + (-1)^h \mu_h \tilde{Q}(\tau), \tau) \prec 0,\ \tau \in [T_{\min}, T_{\max}] \tag{69}$$

which is exactly (61), thus the robust GUAS can be established. □

In comparison with (61), the extension of Condition (a), which has an infinite many decision variables to search, Theorem 2 only has a finite number of decision variable to check the GUAS for event-triggered switched system with ranged sampling intervals. The numerically tractable feature is an obvious advantage over (61) which is a straightforward extension from Condition (a), and this promising feature of Theorem 2 which is actually a generalization of Condition (c) basically benefits from the fact that the system matrices $\tilde{A}_i$ are affine in the corresponding conditions.

5 $\mathcal{L}_2$-Gain Performance Analysis

In the presence of disturbance $\omega(t)$, $\mathcal{L}_2$-gain performance is a disturbance attenuation performance for event-triggered switched system (13)–(15). The basic idea of Condition (a) in Theorem 1, that is abstracting continuous-time system (13)–(15) into a discrete-time version, is difficult to be extended from stability analysis to $\mathcal{L}_2$-gain performance analysis, since the discrete-time abstraction only defines the input-output relation at sampling instants t_k, losing the information over interval (t_k, t_{k+1}). Moreover, the technical difficulties for extension mainly lies in the exponential term $e^{\tilde{A}_i T_s}$. On the other hand, Condition (c) in Theorem 1 can be extended owing to the affineness in system matrix $\tilde{A}_i$. In the following, a numerically tractable result is proposed for $\mathcal{L}_2$-gain performance analysis.

Theorem 3 *Consider event-triggered switched control system (13)–(15), if there exist scalars $M \in \mathbb{N}\backslash\{0\}$, $\mu_h > 0$, $h \in \{1,2\}$, a matrix $\Pi \in \mathbb{M}_d^N$ and symmetric matrices $P_{i,m} \in \mathbb{R}^{2n\times 2n}$, $m \in \{0,\ldots,M\}$, $i \in \mathcal{N}$, such that, for $i \in \mathcal{N}$,*

$$P_{i,m} \succ 0,\ m \in \{0,\ldots,M\} \tag{70}$$

$$\Xi_{i,m,1} \prec 0,\ m \in \{0,\ldots,M-1\} \tag{71}$$

$$\Xi_{i,m,2} \prec 0,\ m \in \{0,\ldots,M-1\} \tag{72}$$

$$\Omega_{i,h} \prec 0,\ h \in \{1,2\} \tag{73}$$

where $\Omega_{i,h} = J_h^\top \sum_{j=1}^{N} \pi_{ji} P_{j,0} J_h - P_{i,M} - (-1)^h \mu_h Q$, and

$$\Xi_{i,m,1} = \begin{bmatrix} \mathscr{D}_1(\tilde{A}_i, P_{i,m+1}, P_{i,m}, T_s/M) & * & * \\ \tilde{E}_i^\top P_{i,m+1} & -\gamma^2 I & * \\ \tilde{C}_i & \tilde{D}_i & -I \end{bmatrix}$$

$$\Xi_{i,m,2} = \begin{bmatrix} \mathscr{D}_2(\tilde{A}_i, P_{i,m+1}, P_{i,m}, T_s/M) & * & * \\ \tilde{E}_i^\top P_{i,m} & -\gamma^2 I & * \\ \tilde{C}_i & \tilde{D}_i & -I \end{bmatrix}$$

then switched system (13)–(15) is GUAS and has an $\mathcal{L}_2$-gain γ under sampled switching rule (17) with $P_i = P_{i,0}$, $i \in \mathcal{N}$.

Proof. The GUAS can be easily obtained by Condition (c) in Theorem 1, thus we focus on the $\mathcal{L}_2$-gain performance in the following. First, we let

$$\Omega(t) = \|y(t)\|^2 - \gamma^2 \|\omega(t)\|^2 \tag{74}$$

and

$$J_k(t) = \int_{t_k^+}^{t} \Omega(s)\mathrm{d}s, \ t \in [t_k^+, t_{k+1}^-] \tag{75}$$

which can imply that

$$J_k(t^-) = \int_{t_k^+}^{t^-} \left(\Omega(s) + \mathcal{D}^+ V_i(\tilde{x}(s))\right) \mathrm{d}s - V_i(\tilde{x}(t^-)) + V_i(\tilde{x}(t_k^+)) \tag{76}$$

where $V_i(\tilde{x}(t))$ is defined as $V_i(\tilde{x}(t)) = \tilde{x}^\top(t) P_i(t) \tilde{x}(t)$, $i \in \mathcal{N}$, with $P_i(t)$, $i \in \mathcal{N}$, defined by (31).

Then, by (75), it can be deduced that $\int_{t_0}^{\infty} \Omega(s)\mathrm{d}s = \sum_{k=0}^{\infty} J_k(t_{k+1}^-)$, which can be rewritten as

$$\int_{t_0}^{\infty} \Omega(s)\mathrm{d}s = \sum\nolimits_{k=0}^{\infty} \int_{t_k^+}^{t_{k+1}^-} \left(\Omega(s) + \mathcal{D}^+ V_i(\tilde{x}(s))\right) \mathrm{d}s + \sum\nolimits_{k=1}^{\infty} \left(V_j(\tilde{x}(t_k^+)) - V_i(\tilde{x}(t_k^-))\right) + V_i(\tilde{x}(t_0)) \tag{77}$$

From (73), one has

$$V_j(\tilde{x}(t_k^+)) - V_i(\tilde{x}(t_k^-)) \le 0, \forall t_k \in \mathcal{S} \tag{78}$$

is satisfied with min-switching rule (17). Moreover, it is obtained that

$$\Omega(t) + \mathcal{D}^+ V_i(\tilde{x}(t)) = \zeta^\top(t) \begin{bmatrix} \Lambda_i & P_i(t)\tilde{E}_i + \tilde{C}_i^\top D_i \\ * & \tilde{D}_i^\top \tilde{D}_i - \gamma^2 I \end{bmatrix} \zeta(t) \tag{79}$$

where $\zeta^\top = [\tilde{x}^\top(t) \ \ \omega^\top(t)]$, $\Lambda_i = \mathscr{D}(\tilde{A}_i, P_i(t)) + \tilde{C}_i^\top \tilde{C}_i$.

Thus, from (71), (72), it gives $\Omega(t) + \mathcal{D}^+ V_i(\tilde{x}(t)) < 0$. Together with (78) and $\tilde{x}(t_0) = 0$, we obtain

$$\int_{t_0}^{\infty} \Omega(s)\mathrm{d}s < 0 \tag{80}$$

which leads to $\int_{t_0}^{\infty} \|y(t)\|^2 \mathrm{d}t \le \gamma^2 \int_{t_0}^{\infty} \|\omega(t)\|^2 \mathrm{d}t$ when $\omega(t) \neq 0$. Therefore, the $\mathcal{L}_2$-gain performance is guaranteed. The proof is complete. □

From Theorem 3, it should be stressed that although the min-switching rule (17) only acts at sampling instants $t_k \in \mathcal{S}$, the $\mathcal{L}_2$-gain level which is defined over $[t_0, \infty)$ can be estimated. This is because (71) and (72) fully characterize the input-output property in the sense of $\mathcal{L}_2$-gain during $[t_k, t_{k+1})$. Moreover, if the robust sampling scheme is considered, the similar extension can be easily made as Theorem 2.

Under the framework of Theorem 3, an estimate of the $\mathcal{L}_2$-gain can be obtained by

$$\begin{gathered} \min \gamma^2 \\ \text{s.t. } (70),\ (71),\ (72),\ (73) \end{gathered} \tag{81}$$

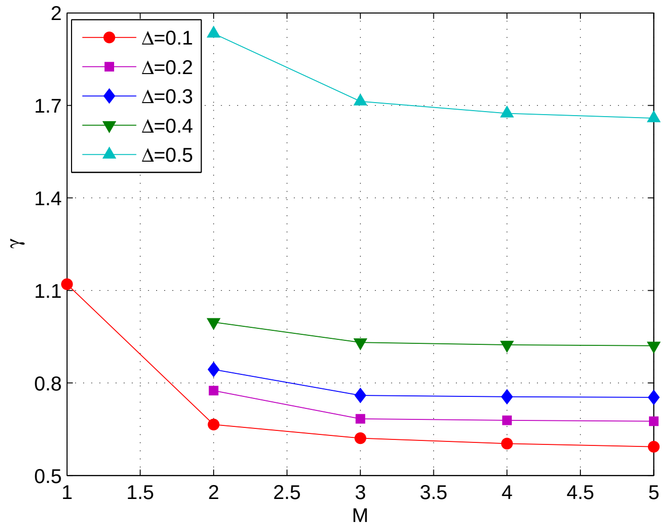

Figure 3: Suboptimal $\mathcal{L}_2$-gain γ with respect to different M

Same as the stability analysis result, the computational results obtained by solving the linear-matrix-inequality-based optimization problems (81) also depend on the choice of M. Less conservative results will be obtained with larger M, at the expense of higher computational cost, which will be shown by the following example.

Example 2 *Consider a switched system with two modes same as in Example 1, and C_i, D_i, E_i, $i \in \{1, 2\}$ are chosen as below:*

$$C_1 = C_2 = [1 \quad 1],\ E_1 = E_2 = \begin{bmatrix} 0.2 \\ 0.5 \end{bmatrix},\ D_1 = D_2 = 0.5 \tag{82}$$

We still consider $\pi_{11} \in [0, 1]$ and $\pi_{12} \in [0, 1]$ with $\pi_{21} = 1 - \pi_{11}$ and $\pi_{22} = 1 - \pi_{12}$, respectively, the increments $\Delta\pi_{11} = 0.1$ and $\Delta\pi_{12} = 0.1$ are taken to divide $[0, 1]$, and search the optimal γ for these discretized points by Theorem 3. The suboptimal $\mathcal{L}_2$-gain is obtained as the minimal value of the optimal γ of all discretized points. Furthermore, given a constant sampling time $T_s = 100$ ms and $\Delta = \{0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5\}$, the suboptimal $\mathcal{L}_2$-gain γ with respect to different M are shown in Figure 3. From Figure 3, it can be observed that the estimated $\mathcal{L}_2$-gain γ decreases as M increases, this is because that a larger M implies a finer division of the sampling interval, and thus a less conservative result can be obtained. Moreover, it can be also found that the control performance becomes worse with a larger state error Δ in event trigger condition, this is consistent with the actual situation. The increasing computational complexities along with M is same as in Table 2, which is not presented here.

6 Conclusions

In this paper, the event-triggered control for switched linear systems has been studied. Three stability criteria are proposed to ensure asymptotic stability of switched system subject to min-switching rule which is only allowed to activate at sampling instants. It has been proved that the three stability criteria are equivalent. Then, taking advantages of one stability criterion with affineness in system matrices, extensions to robust sampling scheme and $\mathcal{L}_2$-gain analysis. In the future work, the controller design, switching rule design and event-triggering condition design should be taken into account based on the stability analysis results proposed in this paper.

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A Proof of Lemma 1

First, given (22) and $0 \prec Y(t) \prec \epsilon I$, $t \in [0, T_s]$, with a sufficiently small $\epsilon > 0$, obviously we can obtain

$$P_m \succ e^{-A^\top \delta_m} P_0 e^{-A\delta_m} - \epsilon \int_0^{\delta_m} e^{-A^\top (\delta_m - t)} e^{-A(\delta_m - t)(t)} \mathrm{d}t \succ 0, \ m \in \{0, \ldots, M\}$$

holds for any initial $P_0 \succ 0$.

Letting $Z_m = \int_0^{\delta_m} e^{-A^\top (\delta_m - t)} Y(t) e^{-A(\delta_m - t)(t)} \mathrm{d}t$ and substituting (22) into $\mathscr{D}_1(A, P_{m+1}, P_m, \delta)$ to get

$$\mathscr{D}_1(A, P_{m+1}, P_m, \delta) = \vartheta_{m,1}(\delta) + \vartheta_{m,2}(\delta) + \vartheta_{m,3}(\delta) \tag{83}$$

where $\delta = T_s/M$ and

$$\begin{aligned} \vartheta_{m,1}(\delta) &= e^{-A^\top \delta_m} \Omega(h) e^{-A\delta_m} \\ \Omega(\delta) &= \mathscr{C}(A, P_0) + \mathscr{E}(A, I, P_0/\delta, P_0/\delta, \delta) \\ \vartheta_{m,2}(\delta) &= -\mathscr{C}(A, Z_m) \\ \vartheta_{m,3}(\delta) &= (Z_m - Z_{m+1})/\delta \end{aligned}$$

Due to $\lim_{\delta \to 0^+} \sup \mathscr{E}(A, I, P_0/\delta, P_0/\delta, \delta) = -\mathscr{C}(A, P_0)$, therefore it yields that $\lim_{\delta \to 0^+} \sup \Omega(\delta) = 0$, which implies

$$\lim_{\delta \to 0^+} \sup \vartheta_{m,1}(\delta) = 0 \tag{84}$$

Moreover, due to $0 \leq \delta_m \leq T_s$, it implies that $e^{-A\delta_m}$ is bounded, $\vartheta_{m,1}(\delta)$ uniformly converges to zero.

In addition, it can be seen that

$$\lim_{\delta \to 0^+} \sup \vartheta_{m,3}(\delta) = -Y(\delta_m) + \mathscr{C}(A, Z_m) \tag{85}$$

which results in

$$\lim_{\delta \to 0^+} \sup(\vartheta_{m,2}(\delta) + \vartheta_{m,3}(\delta)) = -Y(\delta_m) \tag{86}$$

which implies that $\lim_{\delta \to 0^+} \sup(\vartheta_{i,m,2}(\delta) + \vartheta_{m,3}(\delta)) \prec 0$ due to $Y(t) \succ 0$, $t \in [0, T_s]$.

In conclusion, with the aid of (84) and (86), we have

$$\lim_{\delta\to 0^+} \sup \mathscr{D}_1(A, P_{m+1}, P_m, \delta) \prec 0 \tag{87}$$

so there exists a sufficiently small δ_1^* such that

$$\mathscr{D}_1(A, P_{m+1}, P_m, \delta) \prec 0 \tag{88}$$

holds for all $\delta < \delta_1^*$.

By a similar procedure as above, we can consider $\mathscr{D}_2(A, P_{m+1}, P_m, \delta)$ to obtain

$$\lim_{\delta\to 0^+} \sup \mathscr{D}_2(A, P_{m+1}, P_m, \delta) \prec 0 \tag{89}$$

and we can find a sufficiently small δ_2^* such that

$$\mathscr{D}_2(A, P_{m+1}, P_m, \delta) \prec 0 \tag{90}$$

holds for all $\delta < \delta_2^*$.

By setting $\delta^* = \min\{\delta_1^*, \delta_2^*\}$, we can conclude that there exists a sufficiently small δ^* such that (20) and (21) hold for any $\delta < \delta^*$. Due to $\delta = T_s/M$, it is equivalent to the existence of a sufficiently large M^* such that (20) and (21) hold for any $M > M^*$.

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