Robust Exponential Stability and Disturbance Attenuation for Discrete-Time Switched Systems under Arbitrary Switching

Weiming Xiang, Hoang-Dung Tran and Taylor T. Johnson

***Abstract*—In this note, the globally exponential stability of discrete-time switched systems under arbitrary switching is investigated. First, for discrete-time switched nonlinear systems, the globally exponential stability is found to be equivalent to the existence of an M-step sequence with sufficient length and a family of Lyapunov functions, and then a stability criterion is proposed for the nominal linear case in the framework of quadratic Lyapunov function. In order to extend the stability criterion to handle uncertainties, an equivalent condition which has a promising feature that is convex in system matrices is derived, leading to a robust stability criterion for uncertain discrete-time switched linear systems. Moreover, also taking advantage of the convex feature, the disturbance attenuation performance in the sense of ℓ_2-gain is studied. Several numerical examples are provided to illustrate our approach.**

***Index Terms*—Arbitrary switching, ℓ_2-gain, stability, uncertainty, switched system**

I. INTRODUCTION

A switched system is composed of a family of continuous or discrete-time subsystems, described by differential or difference equations, respectively, along with a switching rule governing the switching between the subsystems. The motivation for studying such switched systems comes from the fact that switched systems can be efficiently used to model many practical systems that are inherently multi-modal, thus several dynamical subsystem models are required to describe their behaviors. For example, several real-world cyber-physical systems and industrial processes exhibit switching and hybrid nature intrinsically. Among the large variety of problems studied in theory and encountered in practice, stability analysis of switched systems is a core problem, which attracts considerable research attention in the last decade [1]–[5]. One can study the stability of switched systems with the help of the given the switching rule described by a prescribed state space partitioning [6]–[8] or some known constraints on switching sequence such as dwell time [9]–[13] or average dwell time [14]–[16] restrictions. However, in a number of practical switched systems, the switching sequence is not known *a prior* and stability property has to be examined under arbitrary switching.

The globally exponential stability (GES) of discrete-time switched systems under arbitrary switching is of interest in this paper. In some previous studies, the Lyapunov function approach is a powerful tool for stability analysis of switched systems. As to arbitrary switching, the common Lyapunov function is able to deal with both continuous-time and discrete-time dynamics, e.g., [17]–[19]. Particularly for discrete-time case, an improved result called switched Lyapunov function is developed to significantly reduce the conservativeness in stability analysis [20]. The main aim in this paper is to propose a new Lyapunov function based approach to further reduce the conservatism in stability analysis under arbitrary switching. In this paper, a new stability criterion with less conservativeness over switched Lyapunov function approach will be developed by introducing the conception of M-step sequence listing all possible combinations of subsystems in M steps. First, a general necessary and sufficient condition ensuring the GES for discrete-time switched nonlinear systems is derived. Then, a linear matrix inequality (LMI) based condition is proposed for the case with nominal linear subsystems, which is able to recover the switched Lyapunov function approach by particularly letting $M = 1$ and thus has less conservativeness. However, due to the non-convexity in the system matrices in the LMI conditions, the stability criterion for the nominal case is difficult to be extended to robust stability analysis in the presence of uncertainties, so an alternative equivalent stability criterion expressed by a set of convex condition in system matrices is developed for the sake of extension to robust stability analysis and consequently, a robust stability criterion is derived. Also in virtue of convexity in system matrices, the disturbance attenuation performance in the sense of ℓ_2-gain can be analyzed. Finally, several numerical examples are given in order to emphasize the less conservativeness and effectiveness of the approach.

Notations: $\mathbb{N}$ represents the set of natural numbers, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^+$ is the set of nonnegative real numbers, and $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers, $\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices with real entries. $\|\cdot\|$ stands for Euclidean norm. The notation $A \succ 0$ means A is real symmetric and positive definite. $A \succ B$ means that $A - B \succ 0$. $A^\top$ denotes the transpose of A. In addition, in symmetric block matrices, we use $*$ as an ellipsis for the terms that are induced by symmetry. For two integers k_1 and k_2, $k_1 \leq k_2$, we define $\mathcal{I}[k_1, k_2] \triangleq \{k_1, k_1 + 1, \ldots, k_2\}$.

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II. Preliminaries

In this paper, we consider a discrete-time switched system in the following form

$$\begin{aligned} x(t+1) &= f_{\sigma(t)}(x(t)) \\ x(t_0) &= x_0 \end{aligned} \tag{1}$$

where $x(t), x_0 \in \mathbb{R}^n$ are the system state vector and initial state, respectively. The switching signal σ is defined as $\sigma : \mathbb{N} \to \mathcal{I}[1, N]$, where N is the number of subsystems involved in the switched system. In this paper, no specific restriction is imposed on switching signal σ, namely the arbitrary switching law is considered in the rest of paper. $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in \mathcal{I}[1, N]$, are assumed to be $f_i(0) = 0$ and satisfy the globally Lipschitz condition at $x = 0$, i.e.,

$$\|f_i(x)\| \le \gamma_i \|x\|, \ \forall x \in \mathbb{R}^n, \ \forall i \in \mathcal{I}[1, N] \tag{2}$$

where $\gamma_i > 0$.

The definition of globally exponential stability (GES) for system (1) is given below.

Definition 1: [21] The equilibrium $x = 0$ of system (1) is said to be globally exponential stable (GES) with a decay rate $\mu > 0$ if $\|x(t)\| < ce^{-\mu(t-t_0)} \|x(t_0)\|$ holds for any initial condition $x(t_0) \in \mathbb{R}^n$, any $t \in \mathbb{N}$ and a constant $c > 0$.

Based on a search of a single common quadratic Lyapunov function, the GES can be guaranteed but the results often yield overly conservativeness. As a great improvement in discrete-time domain, the switched Lyapunov function approach which searches for a collection of multiple Lyapunov functions is proposed in [20] to reduce the conservativeness. In this paper, the main aim is to further reduce the conservatism in Lyapunov function based approach for GES analysis of discrete-time switched systems over previous methods. First, we introduce the conception of the M-step sequence, which plays a fundamental role in this paper. The M-step sequence is defined as follows.

Definition 2: For a switched system consisting N subsystem, and given a time window with M-step length, an M-step sequence is a combination of subsystems in M steps. There are N^M combinations of subsystems in M steps, and these N^M combinations are indexed by $\mathcal{I}[1, N^M]$. For the ith sequence of combination in $\mathcal{I}[1, N^M]$, it is expressed by

$$\mathcal{S}_i^M \triangleq \{i_1, i_2, \ldots, i_M\}, \ i_1, \ldots, i_M \in \mathcal{I}[1, N], \ i \in \mathcal{I}[1, N^M]$$

By the M-step sequence, a new M-step switched system consisting of N^M subsystems can be constructed as follows:

$$\begin{aligned} \hat{x}(t+1) &= \mathscr{F}_{\hat{\sigma}(t)}(\hat{x}(t)) \\ \hat{x}(t_0) &= x_0 \end{aligned} \tag{3}$$

where $\mathscr{F}_i \triangleq f_{i_M} \circ \cdots \circ f_{i_2} \circ f_{i_1}$, $i \in \mathcal{I}[1, N^M]$, and $\hat{\sigma}(t)$ denotes the switching among N^M combinations of subsystems, which is also an arbitrary switching signal. Looking into system (1) and (3), since the N^M sequences $\mathcal{S}_i^M$, $i \in \mathcal{I}[1, N^M]$, contain all possible sequences in M steps operation of system (1), $\hat{x}(t)$ actually characterizes all possible evolutions of system (1) state $x(t)$ at every M step, i.e.,

$$\hat{x}(t) = x(tM), \ t \in \mathbb{N} \tag{4}$$

Hence, together with the Lipschitz condition (2), the following result can be obtained.

Lemma 1: Switched system (1) is GES if and only if switched system (3) is GES.

Proof: If system (1) is GES, it implies that $\|x(t)\| < ce^{-\mu(t-t_0)} \|x(t_0)\|$, $\forall t \in \mathbb{N}$. Obviously, given any $M \in \mathbb{N}$, it holds for any $x(tM)$, $t \in \mathbb{N}$, namely, $\|\hat{x}(t)\| < ce^{-\mu(t-t_0)} \|\hat{x}(t_0)\|$, $\forall \hat{t} \in \mathbb{N}$, thus the GES of system (3) can be guaranteed.

On the other hand, if system (3) is GES, which can yield $\|\hat{x}(t)\| < ce^{-\mu(t-t_0)} \|\hat{x}(t_0)\|$, $\forall t \in \mathbb{N}$. In addition, due to the Lipschitz condition (2), and let $\gamma = \max_{i \in \mathcal{I}[1,M]} \prod \gamma_i$, we have $\|x(t)\| \le \gamma \|x(tM)\|$, $t \in [tM+1, (t+1)M]$. Thus, due to $\hat{x}(t_0) = x(t_0)$ and $x(tM) = \hat{x}(t)$, it arrives $\|x(t)\| \le \gamma c e^{-\mu(t-t_0)} \|x(t_0)\|$, and GES of system (1) can be ensured. The proof is complete. ∎

In particular, when the switched linear system is taken into account, system (1) has the following linear form

$$\begin{aligned} x(t+1) &= A_{\sigma(t)} x(t) \\ x(t_0) &= x_0 \end{aligned} \tag{5}$$

where $A_i \in \mathbb{R}^{n \times n}$, $i \in \mathcal{I}[1, N]$. The M-step switched linear system can be derived to

$$\begin{aligned} \hat{x}(t+1) &= \mathscr{A}_{\hat{\sigma}(t)} \hat{x}(t) \\ \hat{x}(t_0) &= x_0 \end{aligned} \tag{6}$$

where $\mathscr{A}_i \triangleq \prod_{m=0}^{M-1} A_{i_{M-m}} \triangleq A_{i_M} \cdots A_{i_2} A_{i_1}$, $i \in \mathcal{I}[1, N^M]$.

In the next section, the stability of switched system (1) will be first analyzed based on Lemma 1 of stability equivalency between system (1) and system (3), and then we will specifically focus on the stability for switched linear system (5).

III. Exponential Stability Analysis for Nominal Switched System

In this section, the M-step sequence conception will be employed to derive a new less conservative stability criteria for switched system (1). First, a general necessary and sufficient condition is proposed based on an M-step sequence and its corresponding Lyapunov functions.

Theorem 1: Switched system (1) is GES if and only if there exist an $M \in \mathbb{N}$, a family of functions $V_i : \mathbb{R}^n \to \mathbb{R}_+$, $i \in \mathcal{I}[1, N^M]$, scalars $0 < \lambda_1 \le \lambda_2$, $\lambda_3 > 0$ such that

$$\lambda_1 \|x\| \le V_i(x) \le \lambda_2 \|x\|, \ \forall x \in \mathbb{R}^n, \ i \in \mathcal{I}[1, N^M] \tag{7}$$

$$\Delta V_{i,j}(x) < -\lambda_3 \|x\|, \ \forall x \in \mathbb{R}^n, \ \forall i, j \in \mathcal{I}[1, N^M] \tag{8}$$

where $\Delta V_{i,j}(x) \triangleq V_j(\mathscr{F}_i(x)) - V_i(x)$.

Proof: Sufficiency: Consider a Lyapunov function for system (3) in the following form

$$V(\hat{x}(t)) = \sum_{i=1}^{N^M} \theta_i(t) V_i(\hat{x}(t))$$

where $\theta_i : \mathbb{N} \to \{0, 1\}$ and $\sum_{i=1}^{N^M} \theta_i(t) = 1$ is the indication function indicating the activated subsystem of system (3).

By (8), it implies that

$$\Delta V(\hat{x}(t)) < -\frac{\lambda_3}{\lambda_2} V(\hat{x}(t)) \Leftrightarrow V(\hat{x}(t+1)) < (1 - \frac{\lambda_3}{\lambda_2}) V(\hat{x}(t))$$

Applying above result yields that

$$V(\hat{x}(t)) < e^{(t-t_0)\ln(1-\frac{\lambda_3}{\lambda_2})}V(\hat{x}(t_0))$$

which leads to

$$\|\hat{x}(t)\| < ce^{-\mu(t-t_0)}\|\hat{x}(t_0)\|$$

where $c = \frac{\lambda_2}{\lambda_1} > 0$, $\mu = -\ln(1-\frac{\lambda_3}{\lambda_2}) > 0$. Thus, GES of system (3) can be established, and therefore system (1) is GES by Lemma 1.

Necessity: Consider any arbitrarily chosen $V_i(x)$, $i \in \mathcal{I}[1, N^M]$, satisfying (7), where M is set to be any number satisfying $M > \frac{\ln(c\lambda_2/\lambda_1)}{\mu}$. If system (1) is GES, we can find the following result

$$\begin{aligned} V_j(x(t+M)) - V_i(x(t)) &< (\lambda_2 ce^{-\mu M} - \lambda_1)\|x(t)\| \\ &= -\lambda_3\|x(t)\| \end{aligned}$$

holds for any $x(t) \in \mathbb{R}^n$, and $\lambda_3 = \lambda_1 - \lambda_2 ce^{-\mu M} > 0$. Furthermore, due to $x(t+M) \in \{x(t+M) \mid x(t+M) = \mathscr{F}_i(x(t)),\ i \in \mathcal{I}[1, N^M]\}$, we have

$$V_j(\mathscr{F}_i(x(t))) - V_i(x(t)) < -\lambda_3\|x(t)\|,\ \forall i, j \in \mathcal{I}[1, N^M]$$

which shows (8) holds. The proof is complete. ■

Remark 1: Theorem 1 proposes a general nonconservative condition for GES of switched system (1). The GES of switched system (1) equals to the existence of an M-step sequence with sufficient length and a collection of Lyapunov functions. Especially, in the proof, it is shown that the non-conservativeness can be achieved provided by a sufficiently large M, which essentially can be quantitatively estimated by $M > \frac{\ln(c\lambda_2/\lambda_1)}{\mu}$. It is interesting to see that, when $M = 1$, conditions (7), (8) are reduced to the classical switched Lyapunov function condition (Theorem 1, [20]). Therefore, it can be concluded that, by introducing the M-step sequence concept, (7), (8) with $M \geq 2$ can further reduce conservativeness of stability analysis result.

However, same like other Lyapunov function based approaches, how to construct appropriate Lyapunov functions is the main challenge for applying Theorem 1. Thus, we particularly consider the linear case of Theorem 1, and the following result can be derived for switched linear system (5).

Theorem 2: For switched linear system (5) and given an $M \in \mathbb{N}$, the following statements are equivalent:

(a) There exist a family of Lyapunov functions $V_i(x) = \sqrt{x^\top P_i x}$, where $P_i \succ 0$, $i \in \mathcal{I}[1, N^M]$, satisfying (7) and (8) proving GES of system (1).

(b) There exist N^M symmetric matrices $P_i \succ 0$, $i \in \mathcal{I}[1, N^M]$ such that the following inequalities hold

$$\mathscr{A}_i^\top P_j \mathscr{A}_i - P_i \prec 0,\ \forall i, j \in \mathcal{I}[1, N^M] \tag{9}$$

The family of Lyapunov functions then are given by

$$V_i(x) = \sqrt{x^\top P_i x},\ i \in \mathcal{I}[1, N^M] \tag{10}$$

Proof: (a) ⇒ (b): Assume there exist a family of Lyapunov functions $V_i(x) = \sqrt{x^\top P_i x}$, where $P_i \succ 0$, $i \in \mathcal{I}[1, N^M]$ satisfying (7) and (8), it is obvious to see that

$$\Delta V_{i,j}(x) = \frac{x^\top(\mathscr{A}_i^\top P_j \mathscr{A}_i - P_i)x}{V_j(\mathscr{A}_i^\top x) + V_i(x)} < 0,\ i, j \in \mathcal{I}[1, N^M]$$

holds for any $x \in \mathbb{R}^n$. Thus, it directly leads to $\mathscr{A}_i^\top P_j \mathscr{A}_i - P_i \prec 0$, i.e., (9) holds.

(b) ⇒ (a): If (10) holds and choose $V_i(x) = \sqrt{x^\top P_i x}$, $i \in \mathcal{I}[1, N^M]$, the λ_1 and λ_2 can be chosen as $\lambda_1 = \sqrt{\lambda_{\min}(P_i)} > 0$ and $\lambda_2 = \sqrt{\lambda_{\max}(P_i)} > 0$, then we have

$$\lambda_1\|x\| \leq V_i(x) \leq \lambda_2\|x\|,\ \forall x \in \mathbb{R}^n,\ i \in \mathcal{I}[1, N^M]$$

Thus, (7) is established.

Then, since (9) holds, there exists an $\epsilon > 0$ such that

$$\mathscr{A}_i^\top P_j \mathscr{A}_i - P_i \prec -\epsilon I,\ i, j \in \mathcal{I}[1, N^M] \tag{11}$$

which yields that

$$x^\top(\mathscr{A}_i^\top P_j \mathscr{A}_i - P_i)x < -\epsilon\|x\|^2,\ \forall x \in \mathbb{R}^n \tag{12}$$

By the definition of $V_i(x)$ according to (10), we have

$$\Delta V_{i,j}(x) = \frac{x^\top(\mathscr{A}_i^\top P_j \mathscr{A}_i - P_i)x}{\sqrt{x^\top \mathscr{A}_i^\top P_j \mathscr{A}_i x} + \sqrt{x^\top P_i x}} \tag{13}$$

Then, let $\rho = \lambda_{\max}(\mathscr{A}_i^\top P_j \mathscr{A}_i) \geq 0$ (due to $P_i \succ 0$, $\forall i \in \mathcal{I}[1, N^M]$), the following inequality can be obtained

$$\Delta V_{i,j}(x) < \frac{-\epsilon\|x\|^2}{\sqrt{\rho}\|x\| + \sqrt{\lambda_2}\|x\|},\ \forall i, j \in \mathcal{I}[1, N^M] \tag{14}$$

Therefore, $\Delta V_{i,j}(x) < -\lambda_3\|x\|$ holds, namely (8) can be established by letting $\lambda_3 = \frac{\epsilon}{\sqrt{\rho}+\sqrt{\lambda_2}} > 0$. The proof is complete. ■

Remark 2: Theorem 2 can be viewed as an improved result over switched Lyapunov function approach for the linear case (Theorem 2, [20]), because the result in [20] is a special case of Theorem 2 with $M = 1$. By Theorem 1 indicating the choice of a larger M leading to a less conservative result, Theorem 2 with $M \geq 2$ is able to yield less conservativeness in stability analysis, which can be shown by a numerical example later.

IV. Robust Stability Analysis for Uncertain Switched System

In this section, the matrices of system (5) are considered to have time-varying uncertainties, which are supposed to belong to the following polytopes

$$A_i \in \mathfrak{A}_i \triangleq \mathbf{co}\{A_i^{[1]}, \ldots, A_i^{[L]},\} \tag{15}$$

where $\mathbf{co}\{\cdot\}$ is the convex-hull operator and $L \in \mathbb{N}$ is the number of vertices of polytope.

However, if $M \geq 2$, the result in Theorem 2 cannot be directly generalized to systems with uncertainties described by (15), due to the presence of intricate multiplication of A_i, i.e., $\mathscr{A}_i = \prod_{m=0}^{M-1} A_{i_{M-m}}$, in condition (9). Thus, we need to turn (9) into a convenient form which can separate A_i from $\mathscr{A}_i$ for the purpose of robust stability analysis. Inspired by the technique used in [22]–[24] for dwell time switching, an equivalent condition of Theorem 2 for arbitrary switching is proposed in the following.

Theorem 3: Consider switched linear system (5), there exist N^M symmetric matrices $P_i \succ 0$, $i \in \mathcal{I}[1, N^M]$ such that (9) holds if and only if there exist MN^M symmetric matrices

$P_{i,m} \succ 0$, $m \in \mathcal{I}[0, M]$, $i \in \mathcal{I}[1, N^M]$ such that the following inequalities hold for $\forall i, j \in \mathcal{I}[1, N^M]$, $\forall m \in \mathcal{I}[1, M]$,

$$A_{i_m}^\top P_{i,m} A_{i_m} - P_{i,m-1} \prec 0 \tag{16}$$

$$P_{i,0} - P_{j,M} \prec 0 \tag{17}$$

then switched linear system (5) is GES.

Proof: Before proving *(9)* $\Leftrightarrow$ *(16), (17)*, we first propose an alternative equivalent condition for (9).

Since $\det(A - sI)$ is a polynomial having a finite number of zeros, so for $|s| \neq 0$ small enough, $A - sI$ is invertible. Thus, (9) holds if and only if

$$P_j - (\mathscr{A}_i - sI)^{-\top} P_i (\mathscr{A}_i - sI)^{-1} \prec 0, \ \forall i, j \in \mathcal{I}[1, N^M] \tag{18}$$

holds with a sufficiently small $|s| \neq 0$.

If there exist P_i, $i \in \mathcal{I}[1, N^M]$ such that (9) holds, we can let $\tilde{P}_i = (\mathscr{A}_i - sI)^{-\top} P_i (\mathscr{A}_i - sI)^{-1} \succ 0$ with a sufficiently small $|s| \neq 0$. Then, substituting $P_i = (\mathscr{A}_i - sI)\tilde{P}_i(\mathscr{A}_i - sI)$ into (18) and swapping i and j imply

$$(\mathscr{A}_i - sI)^\top \tilde{P}_i (\mathscr{A}_i - sI) - \tilde{P}_j \prec 0, \ \forall i, j \in \mathcal{I}[1, N^M]$$

Again because $|s| \neq 0$ could be arbitrarily small, (9) equals to

$$\mathscr{A}_i^\top \tilde{P}_i \mathscr{A}_i - \tilde{P}_j \prec 0, \ \forall i, j \in \mathcal{I}[1, N^M] \tag{19}$$

where $\tilde{P}_i \succ 0$, $i \in \mathcal{I}[1, N^M]$.

Also, it can be said that there exists an $\epsilon > 0$ such that

$$\mathscr{A}_i^\top \tilde{P}_i \mathscr{A}_i - \tilde{P}_j \prec -\epsilon I, \ \forall i, j \in \mathcal{I}[1, N^M] \tag{20}$$

In the following, based on *(9)* $\Leftrightarrow$ *(19)*, we are going to prove *(9)* $\Leftrightarrow$ *(16), (17)* by proving *(19)* $\Leftrightarrow$ *(16), (17)*.

(19) $\Rightarrow$ *(16), (17):* Given any $P_{i,M} \succ 0$, $i \in \mathcal{I}[1, N^M]$, and any $Q_{i,m} \succ 0$, $m \in \mathcal{I}[1, M]$, $i \in \mathcal{I}[1, N^M]$, we can define $P_{i,m}$ in the form of

$$P_{i,m} = \left(\prod_{h=0}^{M-1-m} A_{i_{M-h}}\right)^\top P_{i,M} \left(\prod_{h=0}^{M-1-m} A_{i_{M-h}}\right) + W_{i,m} \tag{21}$$

where

$$W_{i,m} \triangleq \sum_{s=1}^{M-1-m} \left(\prod_{h=s}^{M-1-m} A_{i_{M-h}}\right)^\top Q_{i,M-s} \times \left(\prod_{h=s}^{M-1-m} A_{i_{M-h}}\right) + Q_{i,m}$$

Obviously, we have $P_{i,m} \succ 0$, $m \in \mathcal{I}[1, M]$, $i \in \mathcal{I}[1, N^M]$, and $P_{i,m}$ satisfies

$$A_{i_m}^\top P_{i,m} A_{i_m} - P_{i,m-1} = -Q_{i,m-1}$$

Thus, (16) holds due to $Q_{i,m} \succ 0$, $m \in \mathcal{I}[1, M]$, $i \in \mathcal{I}[1, N^M]$.

Then, letting $m = 0$ in (21), it arrives

$$\left(\prod_{h=0}^{M-1} A_{i_{M-h}}\right)^\top P_{i,M} \left(\prod_{h=0}^{M-1} A_{i_{M-h}}\right) = P_{i,0} - W_{i,0} \tag{22}$$

Since $P_{i,M}$, $i \in \mathcal{I}[1, N^M]$, could be any positive definite matrices, we can choose $P_{i,M} = \tilde{P}_i$, $i \in \mathcal{I}[1, N^M]$, which satisfy (20). Then, using the fact of $\prod_{h=0}^{M-1} A_{i_{M-h}} = \mathscr{A}_i$ and substituting (22) into (20), the following result can be obtained

$$P_{i,0} - P_{j,M} \prec -\epsilon I + W_{i,0} \tag{23}$$

Since $\epsilon > 0$ is fixed and $Q_{i,m} \succ 0$, $m \in \mathcal{I}[1, M]$, $i \in \mathcal{I}[1, N^M]$ can be arbitrarily chosen, $Q_{i,m}$ can be adjusted to be sufficiently small to attain $P_{i,0} - P_{j,M} \prec 0$, i.e., (17) holds.

(16), (17) $\Rightarrow$ *(19):* We consider $\Theta_{i,m} = A_{i_m}^\top P_{i,m} A_{i_m} - P_{i,m-1}$, and if (16) holds, it yields $\Theta_{i,m} \prec 0$ and

$$\begin{aligned}
&\sum_{m=2}^{M} \left(\prod_{h=1}^{m-1} A_{i_{m-h}}\right)^\top \Theta_{i,m} \prod_{h=1}^{m-1} A_{i_{m-h}} + \Theta_{i,1} \\
=&\left(\prod_{h=0}^{M-1} A_{i,M-h}\right)^\top P_{i,M} \prod_{h=0}^{M-1} A_{i,M-h} - P_{i,0} \\
=&\mathscr{A}_i^\top P_{i,M} \mathscr{A}_i - P_{i,0} \preceq 0
\end{aligned}$$

Furthermore, using (17), the following inequality can be derived

$$\mathscr{A}_i^\top P_{i,M} \mathscr{A}_i - P_{j,M} \prec 0$$

Letting $\tilde{P}_i = P_{i,M}$, $i \in \mathcal{I}[1, N^M]$, one can obtain (19) holds. The proof is complete. ■

Remark 3: The following observations can be made for the result in Theorem 3:

1) Theorem 3 provides an alternative stability criterion for switched linear system (5), which is a convexification of Theorem 2 but without introducing any conservativeness. This convex feature will play a crucial role to solve the robust stability analysis problem.
2) Theorem 3 can be viewed as a generalized version of the convex lifted approach from dwell time switching to arbitrary switching. Although the result in [22], [23] can be also applied to arbitrary switching by simply letting dwell time $\tau = 1$, the benefit of lifting idea vanishes in this particular case of $\tau = 1$ and it then reduced to switched Lyapunov function approach in [20]. Taking the advantages of M-step sequence notion proposed in this paper, even for arbitrary switching as dwell time $\tau = 1$, the benefit of M-step sequence idea can work for further reducing conservatism other than conventional switched Lyapunov function method. This will be shown by an example later.
3) Similar as in [22], [23], the cost of the convexification without introducing additional conservativeness is the increase of computation complexity. The computation complexities are listed in Table I.

TABLE I
COMPUTATIONAL COMPLEXITY OF THEOREMS 2 AND 3

	Number of variables	Size of LMIs
Theorem 2	$\frac{n(n+1)N^M}{2}$	nN^{2M}
Theorem 3	$\frac{n(n+1)MN^M}{2}$	$n(N^{2M} + MN^M)$

Based on the convex condition in Theorem 3, the robust exponential stability criterion can be obtained as follow.

Theorem 4: Consider switched linear system (5) with uncertainties described by (15), there exist N^M symmetric matrices $P_i \succ 0$, $i \in \mathcal{I}[1, N^M]$ such that (9) holds if and only if there exist MN^M symmetric matrices $P_{i,m} \succ 0$, $m \in \mathcal{I}[0, M]$, $i \in \mathcal{I}[1, N^M]$ such that the following inequalities hold for $\forall i, j \in \mathcal{I}[1, N^M]$, $\forall m \in \mathcal{I}[1, M]$,

$$(A_{i_m}^{[l]})^\top P_{i,m} A_{i_m}^{[l]} - P_{i,m-1} \prec 0,\ l \in \mathcal{I}[1, L] \tag{24}$$

$$P_{i,0} - P_{j,M} \prec 0 \tag{25}$$

then switched linear system (5) is GES.

Proof: The proof follows from simple convexity arguments, and thus it is omitted here. ■

In this section, based on an alternative equivalent sufficient condition for GES of switched linear system (5), the robust stability analysis problem has been solved in the framework of M-step sequence. The key point is the convexification idea applied to the condition in Theorem 2, which can be further extended to other problems as shown in the next section.

V. Disturbance Attenuation Performance Analysis

The convexification idea involved in Theorem 3 can be not only used for robust stability analysis, it can be also extended to other fundamental problems for switched systems such as disturbance attenuation. Involving exogenous input disturbances $\omega(t) \in \mathbb{R}^l$ and output $y(t) \in \mathbb{R}^p$, we consider the following switched system:

$$\begin{aligned} x(t+1) &= A_{\sigma(t)} x(t) + B_{\sigma(t)} \omega(t) \\ y(t) &= C_{\sigma(t)} x(t) + D_{\sigma(t)} \omega(t) \end{aligned} \tag{26}$$

where $B_i \in \mathbb{R}^{n \times l}$, $C_i \in \mathbb{R}^{p \times n}$, $D_i \in \mathbb{R}^{p \times n}$.

The disturbance attenuation performance of system (26) is considered in the sense of ℓ_2-gain, which means system (26) with $\omega(t) = 0$ is GES and furthermore, under zero initial condition, the input-output relation of system (26) satisfies the following inequality:

$$\sum\nolimits_{t=t_0}^{\infty} y^\top(t) y(t) \le \gamma^2 \sum\nolimits_{t=t_0}^{\infty} \omega^\top(t) \omega(t) \tag{27}$$

where disturbance $\omega(t) \in \ell_2[0, \infty)$.

Theorem 5: Consider system (26), if there exist MN^M symmetric matrices $P_{i,m} \succ 0$, $m \in \mathcal{I}[0, M]$, $i \in \mathcal{I}[1, N^M]$ such that $\forall i, j \in \mathcal{I}[1, N^M]$, $\forall m \in \mathcal{I}[1, M]$,

$$\begin{bmatrix} -P_{i,m-1} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ P_{i,m} A_{i_m} & P_{i,m} B_{i_m} & -P_{i,m} & * \\ C_{i_m} & D_{i_m} & 0 & -I \end{bmatrix} \prec 0 \tag{28}$$

$$P_{i,0} - P_{j,M} \prec 0 \tag{29}$$

then system (26) is GES with $\omega(t) = 0$, and has disturbance attenuation performance in the sense of ℓ_2-gain (27).

Proof: The GES can be easily obtained (28), (29). Hereby, we mainly focus on the disturbance attenuation performance.

Let $J = \sum_{t=0}^{\infty} (y^\top(t) y(t) - \gamma^2 \omega(t) \omega(t))$, and for a sequence $\mathcal{S}_i^M$ in which the start and terminal time are denoted as t_0^i and t_M^i, respectively. Then we define $L_{i,m}(t) = x^\top(t) P_{i,m} x(t)$, $m = t - t_0^i$, and $\Delta L_{i,m}(t) = L_{i,m}(t+1) - L_{i,m}(t)$.

Noting that the initial state $x_0 = 0$, J can be rewritten as

$$\begin{aligned} J &= \sum_{h=0}^{\infty} \left(\sum_{t=hM}^{(h+1)M-1} \Gamma_{i,m}(t) + L_{i,0}(hM) - L_{i,M}((h+1)M) \right) \\ &= \sum_{h=0}^{\infty} \left(\sum_{t=hM}^{(h+1)M-1} \Gamma_{i,m}(t) \right) + \sum_{h=1}^{\infty} (L_{i,0}(hM) - L_{j,M}(hM)) \end{aligned}$$

where $\Gamma_{i,m}(t) = y^\top(t) y(t) - \gamma^2 \omega^\top(t) \omega(t) + \Delta L_{i,m}(t)$. Using Schur complement formula yields

$$\Xi_{i,m} = \begin{bmatrix} \Omega_{i,m} & A_{i_m}^\top P_{i,m} B_{i_m} + C_{i_m}^\top D_{i_m} \\ * & B_{i_m}^\top P_{i,m} B_{i_m} + D_{i_m}^\top D_{i_m} - \gamma^2 I \end{bmatrix} \prec 0$$

where $\Omega_{i,m} = A_{i_m}^\top P_{i,m} A_{i_m} - P_{i,m-1} + C_{i_m}^\top C_{i_m}$. Thus, it leads to $\Gamma_{i,m}(t) < 0$, since $\Gamma_{i,m}(t) = \xi^\top(t) \Xi_{i,m} \xi(t)$, where $\xi(t) = [x^\top(t) \quad \omega^\top(t)]^\top$. Moreover, (29) guarantees $L_{i,0}(hM) - L_{j,M}(hM) < 0, \forall h = 1, 2, \ldots$. Therefore, $J < 0$ can be established, which implies the ℓ_2-gain performance can be established. The proof is complete. ■

Similar as stability analysis, a larger M would lead to a less conservative result at the expense of a higher computational complexity. In ℓ_2-gain performance analysis, it means a larger M yields a smaller γ for the optimization problem below:

$$\min\ \gamma^2 \text{ s.t. } (28), (29) \tag{30}$$

In this section, the M-step sequence approach has been extended to disturbance attenuation performance analysis problem in the sense of ℓ_2-gain. It should be stressed that the extension is made based on Theorem 3 other than Theorem 2, since Theorem 3 has the promising convex feature in system matrices A_i that allows the extension feasible.

VI. Examples

In this section, three examples are provided to show the less conservatism and effectiveness of our proposed approach. Those examples are all executed by using Matlab and toolboxes YALMIP [25] on a personal computer with Windows 7, Intel Core i5-4200U, 1.6GHz, 4 GB RAM.

Example 1: Let us consider the system (5) with matrices $A_i = e^{B_i T}$, where

$$B_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -4 \end{bmatrix} \tag{31}$$

Letting $T = 0.1$, and using switched Lyapunov function approach in [20] (also viewed as $M = 1$ in our M-step sequence approach), it can be found that the LMI problem is not feasible, so that the GES cannot be determined by the approach in [20]. Moreover, by applying the method in [22], [23], the minimum admissible dwell time is computed as 2, which also indicates that the GES of switched system (5) cannot be ascertained for the case of arbitrary switching, for which the minimum dwell time should be 1. However, those results are conservative in stability analysis for the switched system in this example, because GES can be established by the M-step sequence approach proposed in this paper.

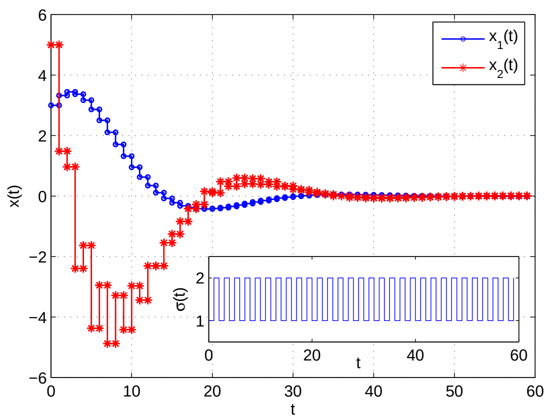

Fig. 1. State response under switching occurring at each time instant.

TABLE II
COMPUTATIONAL TIME (SECOND) BY USING THEOREMS 2 AND 3

	$M=1$	$M=2$	$M=3$	$M=4$	$M=5$	$M=6$	$M=7$
Th.2	infeasible	0.245	0.539	1.344	5.440	17.930	79.227
Th.3	infeasible	0.335	0.929	2.448	7.590	28.043	101.729

Just letting $M = 2$ in the M-step sequence method, the feasibility of the corresponding LMI problems can be established with the following matrices P_i, $i \in \mathcal{I}[1, 2^2]$

$$P_1 = \begin{bmatrix} 2.7228 & 0.1389 \\ 0.1389 & 0.3716 \end{bmatrix}, \; P_2 = \begin{bmatrix} 2.6830 & 0.2773 \\ 0.2773 & 0.3731 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 2.4503 & 0.3519 \\ 0.3519 & 0.3983 \end{bmatrix}, \; P_4 = \begin{bmatrix} 2.8338 & 0.5928 \\ 0.5928 & 0.3593 \end{bmatrix}$$

which is sufficient to guarantee that the system is GES under arbitrary switching.

The convergent state evolution is shown by the following simulation result in Fig. 1, where the extreme switching behavior, i.e., the switching occurs at each time instant, is adopted, and the initial state is assumed to be $x_0 = [3 \;\; 5]^\top$.

Then, In order to show the equivalence between Theorem 2 and Theorem 3, we apply Theorem 3 and the feasibility of LMIs can be established by choosing $M \geq 2$, which is consistent with the result obtained by Theorem 2. Despite the equivalency, the computational cost is different as what has been clarified in Table I. The computational time is listed in Table II. In the rest of simulation, the average computational time is recorded by running the each program 100 times and obtain the average value of them. It can be observed that the computational time grows as M increases, and Theorem 3 uses more time than Theorem 2 for every M, which coincides with Table I.

Example 2: Let us consider the uncertain switched system (5)–(15) with polytopes

$$\mathfrak{A}_1 = \left\{ \begin{bmatrix} 0.952 & 0.094 \\ -0.935 & 0.858 \end{bmatrix}, \begin{bmatrix} 0.152 & -0.306 \\ -1.335 & 0.058 \end{bmatrix} \right\}$$
$$\mathfrak{A}_2 = \left\{ \begin{bmatrix} 0.999 & 0.082 \\ -0.008 & 0.670 \end{bmatrix}, \begin{bmatrix} 0.199 & -0.318 \\ -0.408 & -0.130 \end{bmatrix} \right\}$$

In presence of uncertainties, Theorem 2 only works for the case of $M = 1$, but unfortunately no feasible solution exists for $M = 1$ in Theorem 2, which means the switched Lyapunov function approach in [20] cannot be applied for this uncertain system and, we have to resort to Theorem 4.

Using Theorem 4, we need to increase M in Theorem 4 to check GES. It can be easily found that the GES of the uncertain switched system under arbitrary switching can be guaranteed by choosing $M \geq 3$.

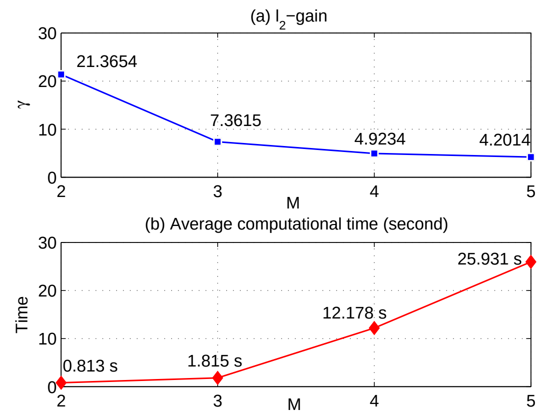

Fig. 2. ℓ_2-gain computation and average computational time.

Example 3: Consider system (26) with two subsystems and $u(t) = 0$, A_i, $i \in \{1, 2\}$ are given same as in Example 1, then let $C_1 = C_2 = [0.1 \;\; 0.5]$, $F_1 = F_2 = [0.1 \;\; 0.2]^\top$ and $G_1 = G_2 = 0$.

First, it needs to be mentioned that the classical switched Lyapunov function approach cannot work for this example in ℓ_2-gain performance analysis, as it equals to the case of $M = 1$ which has no feasible solution. Thus, we increase and use Theorem 5 along with different $M \geq 2$, the minimum ℓ_2-gains by (30) are computed and plotted in Fig. 2(a). Furthermore, the average computational times are also drawn in Fig. 2(b). It can be seen in Fig. 2 that the computed ℓ_2-gain decreases as M increases, which indicates less conservative results can be obtained along with M grows larger, but the price to pay is the computational expenses.

Moreover, it is interesting to see in this example that, the conservativeness is reduced much more obviously when the steps are added for a sequence with less length, as the additional step takes a greater proportion in the new sequence. For example, doubling the value of $M = 1$ to $M = 2$ turns the infeasibility of problem to being feasible, then adding one step from $M = 2$ to $M = 3$ actually extends the length of the old sequence by 1.5 times, and the minimum ℓ_2-gain is reduced significantly, i.e., from 21.3654 to 7.3615. But the latter increment from $M = 3$ to $M = 4$ only produces a little reduction on minimum ℓ_2-gain, just from 4.9234 to 4.2014.

VII. CONCLUSIONS

Globally exponential stability analysis problem for discrete-time switched systems under arbitrary switching has been addressed in this paper. Based on the M-step sequence conception, a necessary and sufficient condition ensuring the GES of switched nonlinear systems is obtained, and then in the framework of quadratic Lyapunov function, a stability criterion for linear case is proposed. In order to extend the result to robust stability analysis in presence of time-varying uncertainties, an equivalent stability criterion with convex feature in system matrices is developed to deal with uncertain switched systems. The disturbance attenuation performance characterized in the sense of ℓ_2-gain is also studied. Finally, numerical examples are given to show the theoretical findings in this paper.

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