Nonconservative Lifted Convex Conditions for Stability of Discrete-Time Switched Systems under Minimum Dwell-Time Constraint

Weiming Xiang, Hoang-Dung Tran and Taylor T. Johnson

***Abstract*—In this note, a novel conception called virtual clock, which is defined by an artificial timer over a finite cycle, is introduced for stability analysis of discrete-time switched linear systems under minimum dwell-time constraint. Two necessary and sufficient conditions associated with a virtual clock with a sufficient length are proposed to ensure the global uniform asymptotic stability of discrete-time switched linear systems. For the two nonconservative stability criteria, the lifted version maintains the convexity in system matrices. Based on the lifted convex conditions, the extensions to ℓ_2-gain computation and $\mathcal{H}_\infty$ control problems are presented in the sequel. In particular, a novel virtual-clock-dependent controller is designed, which outperforms the traditional mode-dependent and common gain controllers. Several numerical examples are provided to illustrate our theoretic results.**

***Index Terms*—Switched system; dwell time; stability; ℓ_2-gain; $\mathcal{H}_\infty$ control**

I. Introduction

A switched system is composed of a finite number of dynamic subsystems described by differential or difference equations, along with a switching rule governing the switching among them. The motivation for studying switched systems comes from the fact that switched systems can be efficiently used to model many practical systems that are inherently multi-modal. In this regard, several dynamic subsystem models are required to describe system behaviors. Among the large variety of problems studied in theory and encountered in practice, stability analysis is one of the core problems in the field of switched systems, which attracts a considerable research attention in the last decade, readers may refer to [1]–[7], and the references cited therein.

One way to study the stability of switched systems is through the notions of dwell time and average dwell time, which are used to characterize the switching rate of a switched system. In the framework of Lyapunov function, a number of methods have been reported and demonstrated to be effective in stability analysis [8]–[15]. Reachability analysis methods have been utilized to solve the stability analysis problems in [16]–[21].

Recently, a promising method called *lifting* approach has been proposed in both continuous-time [22], [23] and discrete-time [24], [25] domains. Those lifted conditions are equivalent convexifications of the well-known results in [10], [11], [26]. This convex feature can significantly facilitate further extensions from stability results to other relevant problems such as robust stability analysis, input-output performance analysis, etc. Some recent generalizations and applications of this lifted idea can be found in terms of more general homogeneous Lyapunov functions [27], extreme cases such as the switched system being fully composed by unstable subsystems [28], hybrid positive systems [29], stochastic systems [30], periodic systems [31], fault-tolerant control [32], etc. In the framework of quadratic Lyapunov functions, those proposed lifted conditions expressed in terms of linear matrix inequalities (LMIs) may sometimes provide tight results, however, they are not necessary in general. Thus, the main problem addressed in this paper arises:

- *Can the necessity for the stability of switched systems be also recovered by further generalizing the lifting approach? Namely, can we derive a* ***nonconservative stability criterion*** *for switched systems in the framework of lifting approach?*

In [33], a nonconservative stability result is derived in the framework of polynomial functions, where the non-conservativeness can be achieved with a sufficiently high degree of the polynomial function. In [34]–[36], the lifting approach is used for stability analysis for switched systems under constrained switching signals and controller design [37], where the switching constraint are mostly characterized by switching graphs or language. In this paper, we will answer this question for a class of time-dependent switched systems under dwell time constraint by gnerealize the results in [24], [25]. To make the generalization and inspired by [34]–[37], we introduce a novel conception called *virtual clock* for discrete-time switched linear systems, which generalizes the framework of dwell time and plays a fundamental role for achieving the non-conservativeness in stability analysis. It also needs to be noted that we further explore the relationship between lifting approach and the well-known method in [11]. With the help of virtual clock, two necessary and sufficient conditions for

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global uniform asymptotic stability of discrete-time switched linear systems under minimum dwell-time constraint are proposed. It is worthwhile mentioning that the nonconservative lifted convex version can be viewed as the extension of the sufficient lifted convex condition [24], and the other one can be also viewed as a nonconservative extension of the well-known result in [11]. Then, by the merit of convexity of lifted conditions, the stability result can be extended to solve ℓ_2-gain computation problem. Finally, based on the virtual clock, a class of $\mathcal{H}_\infty$ virtual-clock-dependent feedback controllers are designed, which have a better performance than the conventional mode-dependent and common gain controllers.

Notations: $\mathbb{N}$ represents the set of natural numbers, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^+$ is the set of nonnegative real numbers, and $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers, $\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices with real entries. The set of $n \times n$ (positive definite) symmetric matrices is denoted by ($\mathbb{S}^n_{\succ 0}$) $\mathbb{S}^n$. $\|\cdot\|$ stands for Euclidean norm. For a set $\mathcal{A}$, $|\mathcal{A}|$ denotes its cardinality. The notation $A \succ 0$ means A is real symmetric and positive definite. $A \succ B$ means that $A - B \succ 0$. $A^\top$ denotes the transpose of A. In addition, in symmetric block matrices, we use $*$ as an ellipsis for the terms that are induced by symmetry. A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$. Moreover, a function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{KL}$ function if, for each fixed s, the function $\beta(r, s)$ is a class $\mathcal{K}$ function with respect to r and, for each fixed r, the function $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. $\mathrm{int}[x]$ denotes the integer part of x. For two integers k_1 and k_2, $k_1 \le k_2$, we define $\mathcal{I}[k_1, k_2] \triangleq \{k_1, k_1 + 1, \ldots, k_2\}$.

II. Preliminaries and Problem Formulation

In this paper, let us consider a class of linear switched systems in the following form

$$\begin{aligned} x(k+1) &= A_{\sigma(k)}x(k) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

where $x(k), x_0 \in \mathbb{R}^n$ are the system state vector and initial state, respectively. The switching signal σ is defined as $\sigma : \mathbb{N} \to \mathcal{I}[1, N]$, where N is the number of subsystems involved in the switched system. Let the switching instants be denoted by k_ℓ, and let $k_0 = 0$ be the initial time by convention. In addition, let $\mathscr{S} \triangleq \{k_\ell\}_{\ell\in\mathbb{N}}$ be the sequence of switching instants. Calling $\mathscr{D}_\tau$ the set of all switching policies with dwell time τ, that is the set of all $\sigma(k)$ for which the time intervals between any successive discontinuities of $\sigma(k)$ satisfy $k_{\ell+1} - k_\ell \ge \tau$, $\forall \ell \in \mathbb{N}$.

Definition 1: [38] The equilibrium $x = 0$ of system (1) is globally uniformly asymptotically stable (GUAS) under switching signal $\sigma(k)$ if, for any initial condition $x(0)$, there exists a class $\mathcal{KL}$ function β such that the solution of system (1) satisfies $\|x(k)\| \le \beta(\|x(0)\|, k)$, $\forall k \in \mathbb{N}$.

One of the basic problems for stability analysis of switched systems is to determine the minimum dwell time guaranteeing the GUAS, named the *minimum dwell time problem* which is defined as

$$\tau_{\min} = \min\{\tau > 0 : \text{System (1) is GUAS } \forall\sigma(k) \in \mathscr{D}_\tau\} \tag{2}$$

For the minimum dwell time problem, two well-known results are recalled.

Theorem 1: [11] Given that for some positive scalar τ, if there exist a collection of symmetric matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i \in \mathcal{I}[1, N]$, such that the following conditions

$$A_i^\top P_i A_i - P_i \prec 0 \tag{3}$$

$$(A_i^\tau)^\top P_j A_i^\tau - P_i \prec 0 \tag{4}$$

hold for all $i, j \in \mathcal{I}[1, N]$, $i \ne j$, then system (1) is GUAS with any switching signals $\sigma(k) \in \mathscr{D}_\tau$.

In terms of Lyapunov functions with quadratic structure, the above result seems to be the best possible so far. However, the LMIs in (4) depend on the exponential term A_i^τ, which is not convex in system matrices A_i. This underlying non-convexity prevents further extensions such as robust stability analysis, controller design, input-output performance analysis, etc. To circumvent this, a set of alternative equivalent conditions in terms of LMIs that are affine in the systems matrices have been proposed in [24], which are called lifted convex conditions.

Theorem 2: [24] Given that for some positive scalar τ, if there exist matrix sequences $P_i : \mathcal{I}[0, \tau] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathcal{I}[1, N]$, such that the following conditions

$$A_i^\top P_i(\tau) A_i - P_i(\tau) \prec 0 \tag{5}$$

$$A_i^\top P_i(k+1) A_i - P_i(k) \prec 0 \tag{6}$$

$$P_i(0) - P_j(\tau) \prec 0 \tag{7}$$

hold for all $i, j \in \mathcal{I}[1, N]$, $i \ne j$ and $k \in \mathcal{I}[0, \tau - 1]$, then system (1) is GUAS with any switching signals $\sigma(k) \in \mathscr{D}_\tau$.

It has been demonstrated that the lifted condition in Theorem 2 is an equivalent convexification of Theorem 1, at an expense of more computational costs in solving LMIs. By the merit of convexity, Theorem 2 can be easily extended to those problems that Theorem 1 is incapable of dealing with, for instance robust stability analysis, controller design and input-output performance analysis. Though Theorem 2 with a set of lifted convex conditions has some obvious advantages over Theorem 1, it is in essence a sufficient stability criterion same as Theorem 1. In this paper, our main aim is to further develop the lifting idea to derive a nonconservative stability criterion for switched system (1).

III. Necessary and Sufficient Lifted Convex Stability Conditions

A. Virtual Clock and Admissible Cycles

The lifted conditions in [24] are expressed in the form of a sequence of inter-dependent LMIs defined over dwell-time interval $[0, \tau]$. In order to further develop the lifting idea, a novel notion we call virtual clock is first introduced.

Definition 2: A virtual clock is defined by

$$\mathscr{C}_L \triangleq \{\theta(k), [0, L-1]\} \tag{8}$$

where $\theta(k)$ is an artificial timer in the form of

$$\theta(k) = k - L\,\mathrm{int}[k/L],\ k \in \mathbb{N} \tag{9}$$

taking values in $[0, L - 1]$.

Given a dwell-time constraint $k_{\ell+1} - k_\ell \geq \tau$, $\forall \ell \in \mathbb{N}$, and a virtual clock $\mathscr{C}_L$ with a length $L \geq \tau$, the conception of admissible switching path with respect to $\{\mathscr{C}_L, \tau\}$ is introduced.

Definition 3: Given a dwell time τ and a virtual clock $\mathscr{C}_L$, $L \geq \tau$, a switching path $\mathcal{S} \triangleq \{i_0, \ldots, i_{L-1}\}$, $i_0, \ldots, i_{L-1} \in \mathcal{I}[1, N]$, over cycle $[0, L-1]$, is an admissible path with respect to $\{\mathscr{C}_L, \tau\}$, if all the switchings in path $\mathcal{S}$ satisfy the dwell-time constraint. The index set of all admissible switching paths is denoted by $\mathscr{A}(L, \tau)$.

Furthermore, the concatenation of two admissible switching paths needs to be considered to fully characterize the evolution of a switching signal. The set of post-admissible paths with respect to an admissible switching path is defined as follows.

Definition 4: Given two admissible switching paths $\mathcal{S}_1 \triangleq \{i_0, \ldots, i_{L-1}\}$ and $\mathcal{S}_2 \triangleq \{j_0, \ldots, j_{L-1}\}$ with respect to $\{\mathscr{C}_L, \tau\}$, $\mathcal{S}_2$ is the post-admissible switching path of $\mathcal{S}_1$, if all the switchings in concatenation path $\{\mathcal{S}_1, \mathcal{S}_2\} \triangleq \{i_0, \ldots, i_{L-1}, j_0, \ldots, j_{L-1}\}$ satisfy the dwell-time constraint. The index set of all post-admissible switching paths of switching path $i \in \mathscr{A}(L, \tau)$ is denoted by $\mathscr{P}\mathscr{A}(i)$, $i \in \mathscr{A}(L, \tau)$.

The following example is presented to illustrate the notions of admissible and post-admissible paths.

Example 1: Consider a switched system with two modes, the dwell time is assumed to be $\tau = 2$, the length of the cycle of the virtual clock is chosen to be $L = 3$. Explicitly, switching paths $\{1, 2, 1\}$ and $\{2, 1, 2\}$ violate the dwell-time constraint since the time between two switchings is 1 which is less than the dwell time 2. By excluding those two inadmissible switching paths, 6 admissible switching paths remain, and they are denoted by $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6\}$, where $\mathcal{S}_1 \triangleq \{1, 1, 1\}$, $\mathcal{S}_2 \triangleq \{1, 1, 2\}$, $\mathcal{S}_3 \triangleq \{1, 2, 2\}$, $\mathcal{S}_4 \triangleq \{2, 1, 1\}$, $\mathcal{S}_5 \triangleq \{2, 2, 1\}$, $\mathcal{S}_6 \triangleq \{2, 2, 2\}$. For the 6 admissible switching paths, the index set is denoted by $\mathscr{A}(3, 2) = \{1, 2, 3, 4, 5, 6\}$

Then, we take $\mathcal{S}_1$ for example to determine the post-admissible switching paths. By excluding the inadmissible post-switching path $\mathcal{S}_4$ since $\{\mathcal{S}_1, \mathcal{S}_4\} \triangleq \{1, 1, 1, 2, 1, 1\}$ obviously violates the dwell-time constraint, the index set of post-admissible switching paths of $\mathcal{S}_1$ is $\mathscr{P}\mathscr{A}(1) = \{1, 2, 3, 5, 6\}$.

By the definitions of admissible path and post-admissible path, we can see that set $\mathscr{A}(L, \tau)$ includes all of the admissible switching paths with dwell-time restriction and $\mathscr{P}\mathscr{A}(i)$, $\forall i \in \mathscr{A}(L, \tau)$ covers all the admissible concatenations of two successive admissible switching paths. Therefore, it can be concluded that all the evolutions of switching signal $\sigma(k)$ under dwell-time switching constraint are involved in sets $\mathscr{A}(L, \tau)$ and $\mathscr{P}\mathscr{A}(i)$, $\forall i \in \mathscr{A}(L, \tau)$.

B. Nonconservative Stability Criterion

Based on the conception of virtual clock, the main contribution, that is the two necessary and sufficient conditions for the stability of discrete-time switched linear system (1), is presented in the following theorem.

Theorem 3: Consider switched system (1), the following three statements are equivalent:

(a) Switched system (1) is GUAS with any switching signals $\sigma(k) \in \mathscr{D}_\tau$;

(b) There exist a scalar $L \geq \tau$ and symmetric matrix sequences $P_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L, \tau)$ such that

$$A_{i_k}^\top P_i(k+1) A_{i_k} - P_i(k) \prec 0, \ i_k \in \mathcal{I}[1, N], \quad i \in \mathscr{A}(L, \tau), \ k = 0, \ldots, L-1 \tag{10}$$

$$P_i(0) - P_j(L) \prec 0, \ i \in \mathscr{P}\mathscr{A}(j), \ j \in \mathscr{A}(L, \tau) \tag{11}$$

(c) There exist a scalar $L \geq \tau$ and symmetric matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L, \tau)$ such that

$$\left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i \left(\prod_{h=1}^{L} A_{i_{L-h}}\right) - P_j \prec 0, \quad i \in \mathscr{P}\mathscr{A}(i), \ j \in \mathscr{A}(L, \tau) \tag{12}$$

where $\prod_{h=1}^{L} A_{i_{L-h}} = A_{i_{L-1}} \cdots A_{i_0}$.

Proof: (a) $\Rightarrow$ (b): Given any $P_i(L) \in \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L, \tau)$, and any $X_i(k) \in \mathbb{S}^n_{\succ 0}$, $k \in \mathcal{I}[0, L-1]$, $i \in \mathscr{A}(L, \tau)$, we can define $P_i(k)$ in the form of

$$P_i(k) = \left(\prod_{h=1}^{L-k} A_{i_{L-h}}\right)^\top P_i(L) \left(\prod_{h=1}^{L-k} A_{i_{L-h}}\right) + Y_i(k) \tag{13}$$

where

$$Y_i(k) \triangleq \sum_{s=1}^{L-k-1} \left(\prod_{h=s}^{L-k-1} A_{i_{L-h}}\right)^\top X_i(L-s) \times \left(\prod_{h=s}^{L-k-1} A_{i_{L-h}}\right) + X_i(k)$$

Obviously, $P_i(k)$ in (13) satisfies $A_{i_k}^\top P_i(k+1) A_{i_k} - P_i(k) = -X_i(k)$. Thus, (10) holds due to $X_i(k) \succ 0$, $k \in \mathcal{I}[0, L-1]$, $i \in \mathscr{A}(L, \tau)$. Then, letting $k = 0$ in (13), it arrives

$$P_i(0) = \left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i(L) \left(\prod_{h=1}^{L} A_{i_{L-h}}\right) + Y_i(0) \tag{14}$$

First, due to $P_i(L) \in \mathbb{S}^n_{\succ 0}$ and $X_i(k) \in \mathbb{S}^n_{\succ 0}$, it is easy to see that $P_i(k) \in \mathbb{S}^n_{\succ 0}$. Then, we consider $P_i(0) - P_j(L)$ in (11), which, using (14), can be rewritten as

$$\begin{aligned} &P_i(0) - P_j(L) \\ &= \left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i(L) \left(\prod_{h=1}^{L} A_{i_{L-h}}\right) + Y_i(0) - P_j(L) \end{aligned} \tag{15}$$

Given any admissible switching path $i \in \mathscr{A}(L, \tau)$, using the definition of state transition matrix $x(L) = \Phi(L, 0) x(0)$, where $\Phi(L, 0) = \prod_{h=1}^{L} A_{i_{L-h}}$, where sequence $\{i_0, \ldots, i_{L-1}\}$ is admissible with respect to dwell-time τ, thus it can be obtained that $P_i(0) - P_j(L) = \Phi^\top(L, 0) P_i(L) \Phi(L, 0) + Y_i(0) - P_j(L)$.

If system (1) is GUAS, there exists a class $\mathcal{KL}$ function β such that $\|x(L)\| \leq \beta(\|x(0)\|, L)$ holds, which means $\|\Phi(L, 0) x(0)\| \leq \beta(\|x(0)\|, L)$. Because β is a class $\mathcal{KL}$ function, it implies that $\lim_{L \to \infty} \beta(\|x(0)\|, L) = 0$ and as a result, one has $\lim_{L \to \infty} \|\Phi(L, 0) x(0)\| = 0$, leading to $\lim_{L\to\infty}\Phi(L,0)=0$. Thus, for any arbitrarily chosen $P_i(L)\in\mathbb{S}^n_{\succ 0}$, there exists an $\epsilon>0$ such that

$$\lim_{L\to\infty}\Phi^\top(L,0)P_i(L)\Phi(L,0)-P_j(L)=-P_j(L)\prec-\epsilon I$$

It implies that there exists a sufficiently large L^* such that, for any $L\geq L^*$, the following inequality holds

$$\Phi^\top(L,0)P_i(L)\Phi(L,0)-P_j(L)\prec-\epsilon I \tag{16}$$

Thus, for any $L\geq L^*$, it has $P_i(0)-P_j(L)\prec-\epsilon I+Y_i(0)$.

Moreover, since $X_i(k)$, $k\in\mathcal{I}[0,L-1]$, $i\in\mathscr{A}(L,\tau)$ are arbitrarily chosen, $X_i(k)$ can be adjusted to make $Y_i(0)$ be sufficiently small to attain $P_i(0)-P_j(L)\prec 0$. Therefore, (11) holds.

(b) $\Rightarrow$ (a): Due to $P_i(k)\in\mathbb{S}^n_{\succ 0}$, for each interval $[nL,(n+1)L]$, we construct Lyapunov functions in the following form

$$V_i(x(k))=\begin{cases}x^\top(k)P_i(\theta(k))x(k), & k\in[nL,(n+1)L-1]\\ x^\top(k)P_i(L)x(k), & k=(n+1)L\end{cases} \tag{17}$$

where $i\in\mathscr{A}[L,\tau]$ and $\theta(k)$ is the artificial timer defined by (9). Then, for any admissible switching path $i\in\mathscr{A}[L,\tau]$, we have

$$\begin{aligned}\Delta V_i(x(k))&=V_i(x(k+1))-V_i(x(k))\\&=x^\top(k)[A_{i_k}^\top P_i(\theta(k+1))A_{i_k}-P_i(\theta(k))]x(k)\end{aligned}$$

and (10) ensures that

$$\Delta V_i(x(k))<0,\ k\in[nL,(n+1)L-1],\ n\in\mathbb{N} \tag{18}$$

Furthermore, let us consider the concatenation between two switching paths. Suppose $j\in\mathscr{A}(L,\tau)$ and $i\in\mathscr{PA}(j)$, (11) implies that

$$V_i(x(k))<V_j(x(k)),\ k=(n+1)L,\ n\in\mathbb{N} \tag{19}$$

From (18) and (19), the value of Lyapunov function (17) is always decreasing along with the time. Therefore, the GUAS of system (1) can be established by standard Lyapunov theorem [38].

(b) $\Rightarrow$ (c): Since (11) holds, using (15) implies that the following inequality can be obtained

$$\left(\prod_{h=1}^{L}A_{i_{L-h}}\right)^\top P_i(L)\left(\prod_{h=1}^{L}A_{i_{L-h}}\right)-P_j(L)\prec-Y_i(0)$$

Moreover, by $Y_i(k)$ defined in (13), it can be obtained $Y_i(0)\succ 0$, thus we have

$$\left(\prod_{h=1}^{L}A_{i_{L-h}}\right)^\top P_i(L)\left(\prod_{h=1}^{L}A_{i_{L-h}}\right)-P_j(L)\prec 0$$

which means (12) holds, just by letting $P_i=P_i(L)$.

(c) $\Rightarrow$ (b): (10) and $P_i(k)\in\mathbb{S}^n_{\succ 0}$ have been established at the beginning of (a) $\Rightarrow$ (b), so in the rest we only need to consider (11). Since (12) holds, it implies that there exists an $\epsilon>0$ such that

$$\left(\prod_{h=1}^{L}A_{i_{L-h}}\right)^\top P_i\left(\prod_{h=1}^{L}A_{i_{L-h}}\right)-P_j\prec-\epsilon I$$

TABLE I
COMPUTATIONAL COMPLEXITY OF STATEMENTS (a) AND (b)
($M=|\mathscr{A}[L,\tau]|,m(i)=|\mathscr{PA}(i)|,i\in\mathscr{A}[L,\tau]$)

Statement	Number of variables	Size of LMIs
(b)	$\frac{n(n+1)(L+1)M}{2}$	$nM(L-1)+n\sum_{i\in\mathcal{I}[1,M]}m(i)$
(c)	$\frac{n(n+1)M}{2}$	$n\sum_{i\in\mathcal{I}[1,M]}m(i)$

Again, letting $P_i(L)=P_i$ and using (15) can derive the following inequality $P_i(0)-P_j(L)\prec-\epsilon I+Y_i(0)$. Since $X_i(k)$, $k\in\mathcal{I}[0,L-1]$, $i\in\mathscr{A}(L,\tau)$ are arbitrarily chosen, $X_i(k)$ can be adjusted to make $Y_i(0)$ sufficiently small to achieve $P_i(0)-P_j(L)\prec 0$. Therefore, (11) holds. ■

Remark 1: Theorem 3 is the main result in this paper, some observations are made as below:

(1) Statement (b) generalizes the lifted convex idea of Theorem 2 on the basis of virtual clock. Unlike in [24] where the lifted idea was implemented over the dwell-time interval, Statement (b) is derived based on the cycle of virtual clock. Moreover, it should be stressed that this generalization is able to finally achieve a nonconservative stability criterion, if the system is equipped with a virtual clock with a sufficiently long cycle.

(2) Statement (c) can be viewed as an improvement of Theorem 1 proposed in [11], since the condition in Statement (c) is a necessary and sufficient condition in contrast to Theorem 1 which is only a sufficient one. However, similar as in Theorem 1, system matrices A_i are not convex in (12), due to the presence of intricate multiplication of A_i, that is, $\prod_{h=1}^{L}A_{i_{L-h}}$.

(3) Like the relationship between Theorems 1 and 2, Statement (b) is an equivalent convexification of Statement (c). Despite the equivalence of two conditions, the computational complexities are different. The computational complexities are listed in Table I. In Table I, it is shown that Statement (b) has a higher computational complexity than Statement (c) for checking the stability of system (1), which is the cost of convexification while still maintaining the same conservativeness in stability analysis.

If the special case $\tau=1$, that is the arbitrary switching case, is taken into account, it leads to $|\mathscr{A}(L,\tau)|=N^L$, so that the following corollary can be derived.

Corollary 1: Consider switched system (1) under arbitrary switching, the following three statements are equivalent:

(a) Switched system (1) is GUAS;

(b) There exist a scalar $L\geq\tau$ and symmetric matrix sequences $P_i:\mathcal{I}[0,L]\to\mathbb{S}^n_{\succ 0}$, $i\in\mathcal{I}[1,N^L]$, such that

$$A_{i_k}^\top P_i(k+1)A_{i_k}-P_i(k)\prec 0,\ i_k\in\mathcal{I}[1,N] \tag{20}$$

$$P_i(0)-P_j(L)\prec 0 \tag{21}$$

hold for all $i,j\in\mathcal{I}[1,N^L]$ and $k=0,\ldots,L-1$;

(c) There exist a scalar $L\geq\tau$ and symmetric matrices $P_i\in\mathbb{S}^n_{\succ 0}$, $i\in\mathcal{I}[1,N^L]$, such that

$$\left(\prod_{h=1}^{L}A_{i_{L-h}}\right)^\top P_i\left(\prod_{h=1}^{L}A_{i_{L-h}}\right)-P_j\prec 0 \tag{22}$$

hold for all $i, j \in \mathcal{I}[1, N^L]$.

Proof: For arbitrary switching, the admissible cycle allows all possible switching paths, so it has $\mathscr{A}(L,\tau) = \{1, 2, \dots, N^L\}$ and $\mathscr{P}\mathscr{A}(i) = \{1, 2, \dots, N^L\}, \forall i \in \mathscr{A}(L,\tau)$. Thus, it can be proved based on Theorem 3 and is omitted. ■

In this subsection, two necessary and sufficient conditions are provided to ensure the GUAS of discrete-time switched linear systems. The following numerical example is provided to illustrate the improvement made by virtual clock approach over other approaches.

Example 2: Let us consider the system (1) with two subsystems as below:

$$A_1 = \begin{bmatrix} 0.969 & 0.0761 \\ -0.7607 & 0.8929 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.9997 & 0.0685 \\ -0.0068 & 0.7259 \end{bmatrix}$$

In order to show the improvement made by our approach, we first use the approaches in [11], [24], namely Theorems 1 and 2, to compute the minimum dwell time $\tau_{\min} = 3$, and meanwhile we apply the virtual clock approach. The computation results are shown in Table II.

TABLE II
COMPUTATION ON MINIMAL DWELL TIME $\tau_{\min}$

Methods	$\tau_{\min}$	Computational Time
(b) in Corollary 1 $(L = 3)$	1	0.101412 seconds
(c) in Corollary 1 $(L = 3)$	1	0.04883 seconds

It can be found that less conservative results can be obtained when we let $L = 3$, that is $\tau_{\min} = 1$. Moreover, note that $\tau_{\min} = 1$ indicates that the system is GUAS under arbitrary switching, and this is explicitly a nonconserevative result for the proposed switched system. The convergent state evolutions are shown by the simulation results in Figure 1, where 1000 state trajectories are randomly generated. All these state trajectories converge to the origin to show the GUAS of the system. Furthermore, assuming the initial state is $x(0) = [1\ 1]^\top$, the Lyapunov function derived from condition (b) is strictly decreasing as shown in Figure 1, which guarantees the GUAS.

Though Statements (b) and (c) can both achieve the same result, the computational complexities are different as Table I indicates. The computational time is also listed in Table II, it shows that Statement (b) needs to afford a higher computational cost, which is consistent with Table I.

According to Table I, the number of variables and LMIs will grow to a large number as L is a large number, this may lead to difficulties in practical use of the developed approach. However, if the information of the switching rule is known, the computation cost will be affordable even L is large. Let us consider the example in [18], the system matrices are

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 0.1 \\ -0.9 & 0.9 \end{bmatrix}$$

The switching is a periodic one with $k_{\ell+1} - k_\ell = 15$, $\forall \ell \in \mathbb{N}$ and $\delta(0) = 1$. Thus, we can choose $L = 15$ and two admissible cycles $\mathcal{S}_1 = \{1, 1, \dots, 1\}$, $\mathcal{S}_2 = \{2, 2, \dots, 2\}$ are sufficient to characterize the switching law and corresponding virtual clock. Applying both conditions (b) and (c) in Theorem 3, the stability can be established with $\tau = 15$, which is consistent with [18]. Moreover, it can be seen that $\tau_1 5$ is a nonconservative result since the spectrum of $A_1^{14} A_2^{14}$ contains one eigenvalue outside the unit disc.

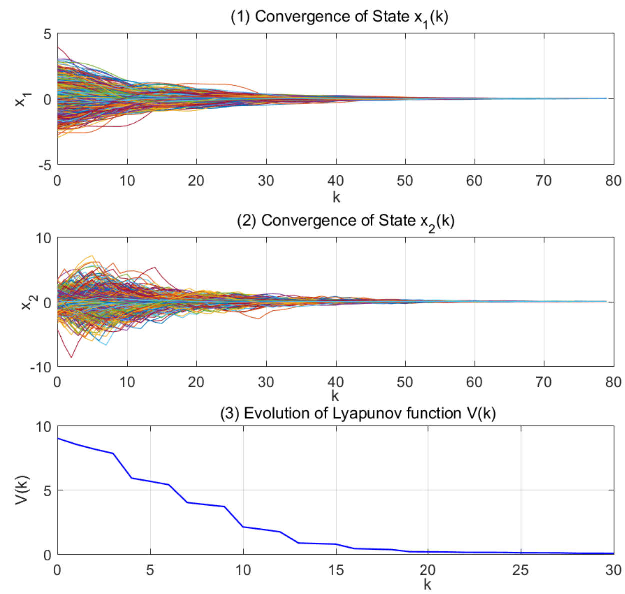

Fig. 1. 1000 randomly generated state trajectories and evolution of Lyapunov function.

IV. ℓ_2-INDUCED GAIN COMPUTATION

Involving control input $u(k) \in \mathbb{R}^m$, exogenous input disturbances $\omega(k) \in \mathbb{R}^l$ and output $y(k) \in \mathbb{R}^p$, we consider the following switched system in the rest of this paper:

$$\begin{aligned} x(k+1) &= A_{\sigma(k)} x(k) + B_{\sigma(k)} u(k) + E_{\sigma(k)} \omega(k) \\ y(k) &= C_{\sigma(k)} x(k) + D_{\sigma(k)} u(k) + F_{\sigma(k)} \omega(k) \end{aligned} \tag{23}$$

where B_i, C_i, D_i, E_i and F_i are constant matrices with appropriate dimensions.

Definition 5: For $\gamma > 0$, system (23) with $u(k) = 0$ is said to be GUAS with an ℓ_2-gain performance at a level γ, if system (23) is GUAS when $u(k) = 0$ and $\omega(k) = 0$, and under zero initial conditions, the following inequality holds for all nonzero $\omega(k) \in \ell_2[0, \infty)$,

$$\sum\nolimits_{k=0}^{\infty} y^\top(k) y(k) \le \gamma^2 \sum\nolimits_{k=0}^{\infty} \omega^\top(k)\omega(k) \tag{24}$$

where γ is called the ℓ_2-gain, and the ℓ_2-induced gain of system (23) with $u(k) = 0$ is defined by $\gamma^* \triangleq \inf\{\gamma \ge 0 :$ (24) holds, $\forall \omega(k) \in \ell_2[0,\infty),\ \omega(k) \neq 0\}$.

Based on the virtual clock idea, the following theorem can be derived for ℓ_2-gain performance analysis of system (23).

Theorem 4: Given a scalar $\gamma > 0$ and consider system (23) with $u(k) = 0$ and $\sigma(k) \in \mathscr{D}_\tau$, if there exist a scalar $L \ge \tau$ and symmetric matrix sequences $P_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L,\tau)$ such that

$$\Theta_i(k) \prec 0,\ i \in \mathscr{A}(L,\tau),\ k = 0, \dots, L-1 \tag{25}$$

$$P_i(0) - P_j(L) \prec 0,\ i \in \mathscr{P}\mathscr{A}(j),\ j \in \mathscr{A}(L,\tau) \tag{26}$$

where

$$\Theta_i(k) = \begin{bmatrix} -P_i(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ P_i(k+1)A_{i_k} & P_i(k+1)E_{i_k} & -P_i(k+1) & * \\ C_{i_k} & F_{i_k} & 0 & -I \end{bmatrix}$$

and $i_k \in \mathcal{I}[1, N]$, then system (23) with $u(k) = 0$ and $\omega(k) = 0$ is GUAS, and has an ℓ_2-gain γ.

Proof: The GUAS can be obtained by (25), (26), directly using (10) and (11) in Theorem 3. In the rest of proof, we therefore only need to focus on proving ℓ_2-gain performance. Let

$$J = \sum\nolimits_{k=0}^{\infty} (y^\top(k)y(k) - \gamma^2\omega(k)\omega(k)) \tag{27}$$

Then, we construct Lyapunov function $V_i(x(k))$ in the form of (17). Noting that the initial state $x_0 = 0$, J can be rewritten as

$$\begin{aligned} J &= \sum_{h=0}^{\infty} \left(\sum_{k=hL}^{(h+1)L-1} \Gamma_i(k) + V_i(x(hL)) - V_i(x((h+1)L)) \right) \\ &= \sum_{h=0}^{\infty} \left(\sum_{k=hL}^{(h+1)L-1} \Gamma_i(k) \right) + \sum_{h=1}^{\infty} (V_i(x(hL)) - V_j(x(hL))) \end{aligned}$$

where $\Gamma_i(k) = y^\top(k)y(k) - \gamma^2\omega^\top(k)\omega(k) + \Delta V_i(x(k))$. Using Schur complement formula, (25) yields

$$\Xi_i(k) = \begin{bmatrix} \Omega_i(k) & A_{i_k}^\top P_i(k+1)E_{i_k} + C_{i_k}^\top F_{i_k} \\ * & E_{i_k}^\top P_i(k+1)E_{i_k} + F_{i_k}^\top F_{i_k} - \gamma^2 I \end{bmatrix} \prec 0$$

where $\Omega_i(k) = A_{i_k}^\top P_i(k+1)A_{i_k} - P_i(k) + C_{i_k}^\top C_{i_k}$. Thus, it leads to $\Gamma_i(k) < 0$, since $\Gamma_i(k) = \xi^\top(k)\Xi_i(k)\xi(k)$, where $\xi(k) = [x^\top(k) \quad \omega^\top(k)]^\top$. Moreover, (26) guarantees $V_i(x(hL)) - V_j(x(hL)) < 0, \forall h = 1, 2, \ldots$. Therefore, $J < 0$ can be established, which implies the ℓ_2-gain performance can be established. The proof is complete. ∎

Remark 2: Similar as stability analysis, a larger L would lead to a less conservative result at an expense of a higher computational complexity. Regarding ℓ_2-gain performance analysis, it means that a larger L will yield a smaller γ for the optimization problem below:

$$\min \ \gamma^2 \text{ s.t. } (25), (26) \tag{28}$$

Example 3: Consider system (23) with two subsystems and $u(k) = 0$, A_i, $i \in \{1, 2\}$ are given same as in Example 2, then let $C_1 = C_2 = [0.1 \quad 0.2]$, $E_1 = E_2 = [0.1 \quad -0.1]^\top$ and $F_1 = F_2 = 0$.

In Example 2, the GUAS of the proposed system only can be established for $\tau_{\min} \geq 3$. As an extension of stability analysis, the ℓ_2-gain computation result in [24] is unable to compute ℓ_2-gain for $\tau_{\min} < 3$.

Using Theorem 4, we can compute the ℓ_2-induced gain for $\tau_{\min} < 3$. For example with $\tau_{\min} = 1$, the ℓ_2-induced gain can be computed with $L \geq 3$, , see Table IV for the the ℓ_2-induced gains with different L. It can be observed that the estimation of ℓ_2-induced gain decreases as L grows, but more computational time is required for the computation.

TABLE III

ℓ_2-INDUCED GAIN COMPUTATION AND COMPUTATIONAL TIME (SECOND)

	$L = 3$	$L = 4$	$L = 5$
ℓ_2-induced gain γ^*	7.7453	3.9781	3.9669
Computational time	0.376973	3.9781	16.938713

V. VIRTUAL-CLOCK-DEPENDENT $\mathcal{H}_\infty$ CONTROL

In this section, the $\mathcal{H}_\infty$ control problem for system (23) is considered. Taking advantage of the convex feature in the virtual clock idea, a novel virtual-clock-dependent controller is introduced in the form of

$$u(k) = K_i(\theta(k))x(k), \ i \in \mathcal{I}[1, N] \tag{29}$$

where $\theta(k)$ is the artificial timer defined by (9). Substituting controller (29) into system (23), the closed-loop system becomes

$$\begin{aligned} x(k+1) &= \tilde{A}_{\sigma(k)}x(k) + E_{\sigma(k)}\omega(k) \\ y(k) &= \tilde{C}_{\sigma(k)}x(k) + F_{\sigma(k)}\omega(k) \end{aligned} \tag{30}$$

where $\tilde{A}_i = A_i + B_iK_i(\theta(k))$, $\tilde{C}_i = C_i + D_iK_i(\theta(k))$. The design objective is to find proper feedback gains $K_i(\theta(k))$, $i \in \mathcal{I}[1, N]$, to ensure the ℓ_2-gain performance of closed-loop system (30).

Theorem 5: Given a scalar $\gamma > 0$ and consider system (23) with $\sigma(k) \in \mathscr{D}_\tau$, if there exist a scalar $L \geq \tau$ and symmetric matrix sequences $Q_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L, \tau)$, matrix sequences $U_i : \mathcal{I}[0, L-1] \to \mathbb{R}^{m \times n}$, $W_i : \mathcal{I}[0, L-1] \to \mathbb{R}^{n \times n}$, $i \in \mathcal{I}[1, N]$ such that

$$\Xi_i(k) \prec 0, \ i \in \mathscr{A}(L, \tau), \ k = 0, \ldots, L-1 \tag{31}$$

$$Q_j(L) - Q_i(0) \prec 0, \ i \in \mathscr{PA}(j), \ j \in \mathscr{A}(L, \tau) \tag{32}$$

where

$$\Xi_i(k) = \begin{bmatrix} \Xi_{i,1}(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \Xi_{i,2}(k) & E_{i_k} & -Q_i(k+1) & * \\ \Xi_{i,3}(k) & F_{i_k} & 0 & -I \end{bmatrix}$$

in which $\Xi_{i,1}(k) = Q_i(k) - W_i^\top(k) - W_i(k)$, $\Xi_{i,2}(k) = A_{i_k}W_i(k) + B_{i_k}U_i(k)$ and $\Xi_{i,3}(k) = C_{i_k}W_i(k) + D_{i_k}U_i(k)$, $i_k \in \mathcal{I}[1, N]$, then closed-loop system (30) is GUAS with $\omega(k) = 0$ and has an ℓ_2-gain γ, where virtual-clock dependent feedback gains $K_i(\theta(k))$, $i \in \mathcal{I}[1, N]$ are

$$K_i(\theta(k)) = U_i(\theta(k))W_i^{-1}(\theta(k)), \ i \in \mathcal{I}[1, N] \tag{33}$$

and $\theta(k)$ is the timer defined by (9).

Proof: By feedback gains (33) and timer (9), we have

$$W_i(k) = K_i(k)U_i(k), \ k = 0, \ldots, L-1, \ i \in \mathcal{I}[1, N] \tag{34}$$

Substituting (34) into (31), it arrives

$$\begin{bmatrix} \Xi_{i,1}(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \tilde{A}_{i_k}W_i(k) & E_{i_k} & -Q_i(k+1) & * \\ \tilde{C}_{i_k}W_i(k) & F_{i_k} & 0 & -I \end{bmatrix} \prec 0 \tag{35}$$

In addition, by (31), it yields

$$Q_i(k) - W_i^\top(k) - W_i(k) \prec 0 \tag{36}$$

which means that $W_i(k)$ is of full rank. Moreover, as $Q_i(k)$ is strictly positive definite, we also have

$$(Q_i(k) - W_i(k))^\top Q_i^{-1}(k)(Q_i(k) - W_i(k)) \succeq 0 \quad (37)$$

which is equivalent to

$$W_i^\top(k)Q_i^{-1}(k)W_i(k) \succeq W_i^\top(k) + W_i(k) - Q_i(k) \quad (38)$$

It follows that

$$\begin{bmatrix} -W_i^\top(k)Q_i^{-1}(k)W_i(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \tilde{A}_{i_k}W_i(k) & E_{i_k} & -Q_i(k+1) & * \\ \tilde{C}_{i_k}W_i(k) & F_{i_k} & 0 & -I \end{bmatrix} \prec 0$$

which equals to

$$\begin{bmatrix} -Q_i^{-1}(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \tilde{A}_{i_k} & E_{i_k} & -Q_i(k+1) & * \\ \tilde{C}_{i_k} & F_{i_k} & 0 & -I \end{bmatrix} \prec 0 \quad (39)$$

Pre- and post-multiplying $\operatorname{diag}\{I, I, Q^{-1}(k+1), I\}$ and letting $P_i(k) = Q_i^{-1}(k)$, it can be equivalently expressed as

$$\begin{bmatrix} -P_i(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ P_i(k+1)\tilde{A}_{i_k} & P_i(k+1)E_{i_k} & -P_i(k+1) & * \\ \tilde{C}_{i_k} & F_{i_k} & 0 & -I \end{bmatrix} \prec 0$$

Moreover, (32) leads to $P_i(0) - P_j(L) \prec 0$, $i \in \mathscr{PA}(j)$, $j \in \mathscr{A}(L,\tau)$. Therefore, by Theorem 4, closed-loop system (30) is GUAS with $\omega(k) = 0$ and has an ℓ_2-gain γ. ■

Remark 3: In Theorem 5, the designed controller gains are both mode-dependent and time-dependent, that is virtual-clock-dependent. The feedback gain at each instant is chosen not only based on the activated mode, but also dependent on the timer $\theta(k)$. For each subsystem, the controller gain actually has L values for selection for different steps in an L-step sequence. Thus, in order to implement this virtual-clock-dependent controller, a virtual clock $\mathscr{C}_L$ needs to be equipped to monitor the value of timer $\theta(k)$ to select the proper gain.

Two special cases are considered in the sequel. First, if we only consider constant mode-dependent feedback gains for subsystems, the following corollary can be derived.

Corollary 2: Given a scalar $\gamma > 0$ and consider system (23) with $\sigma(k) \in \mathscr{D}_\tau$, if there exist a scalar $L \geq \tau$ and symmetric matrix sequences $Q_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L,\tau)$, matrices $U_i \in \mathbb{R}^{m\times n}$, $W_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{I}[1,N]$ such that

$$\Xi_i(k) \prec 0,\ i \in \mathscr{A}(L,\tau),\ k = 0, \dots, L-1 \quad (40)$$

$$Q_j(L) - Q_i(0) \prec 0,\ i \in \mathscr{PA}(j),\ j \in \mathscr{A}(L,\tau) \quad (41)$$

where

$$\Xi_i(k) = \begin{bmatrix} Q_i(k) - W_i^\top - W_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{i_k}W_i + B_{i_k}U_i & E_{i_k} & -Q_i(k+1) & * \\ C_{i_k}W_i + D_{i_k}U_i & F_{i_k} & 0 & -I \end{bmatrix}$$

then closed-loop system (30) is GUAS with $\omega(k) = 0$ and has an ℓ_2-gain γ, where mode-dependent feedback gains K_i, $i \in \mathcal{I}[1,N]$ are

$$K_i = U_i W_i^{-1},\ i \in \mathcal{I}[1,N] \quad (42)$$

TABLE IV

ℓ_2-INDUCED GAINS OF THE CLOSED-LOOP SYSTEM (C.G.: COMMON GAIN METHOD; M.D.: MODE-DEPENDENT METHOD; V.C.: VIRTUAL-CLOCK METHOD)

	C.G.	M.D.	V.C. ($L=2$)	V.C. ($L=3$)	V.C. ($L=4$)	V.C. ($L=5$)
γ^*	3.2586	3.0047	3.0041	2.9936	2.9914	2.9908

Proof: Just let $W_i(k) = W_i$, $U_i(k) = U_i$, $\forall k \in \mathcal{I}[1, L]$, in Theorem 5, to complete the proof. ■

Lastly, if the switching signal cannot be detected online which means $\sigma(k)$ is not available, a common feedback control gain valid for all modes has to be designed.

Corollary 3: Given a scalar $\gamma > 0$ and consider system (23) with $\sigma(k) \in \mathscr{D}_\tau$, if there exist a scalar $L \geq \tau$ and symmetric matrix sequences $Q_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L,\tau)$, matrices $U \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{n\times n}$ such that

$$\Xi_i(k) \prec 0,\ i \in \mathscr{A}(L,\tau),\ k = 0, \dots, L-1 \quad (43)$$

$$Q_j(L) - Q_i(0) \prec 0,\ i \in \mathscr{PA}(j),\ j \in \mathscr{A}(L,\tau) \quad (44)$$

where

$$\Xi_i(k) = \begin{bmatrix} Q_i(k) - W^\top - W & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{i_k}W + B_{i_k}U & E_{i_k} & -Q_i(k+1) & * \\ C_{i_k}W + D_{i_k}U & F_{i_k} & 0 & -I \end{bmatrix}$$

then closed-loop system (30) is GUAS with $\omega(k) = 0$ and has an ℓ_2-gain γ, where common feedback gain K are

$$K = UW^{-1} \quad (45)$$

Proof: Just let $W = W_i$, $U = U_i$, $\forall i \in \mathcal{I}[1, N]$, in Corollary 2, to complete the proof. ■

Example 4: Consider switched system (23) with the following system matrices:

$$A_1 = \begin{bmatrix} -0.92 & -0.34 \\ 1.0350 & -0.31 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.92 & -0.34 \\ 1.0350 & -0.31 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.52 \\ 0.40 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.85 \\ 0.05 \end{bmatrix},\ C_1^\top = \begin{bmatrix} -0.49 \\ 0.34 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0.67 & -0.42 \end{bmatrix},\ D_1 = 1.44,\ D_2 = -0.36$$

$$E_1 = \begin{bmatrix} 0.90 \\ 0.97 \end{bmatrix},\ E_2 = \begin{bmatrix} 0.06 \\ -0.08 \end{bmatrix},\ F_1 = F_2 = 0$$

The minimum dwell time is assumed to be $\tau_{\min} = 1$. In order to show the advantages of virtual clock approach, Theorem 5, Corollary 2 and Corollary 3 are used to design virtual clock-dependent, mode-dependent and common feedback controller respectively to minimize the ℓ_2-gain of the closed-loop system. The computation results are shown in Table IV. In Table IV, it can be seen that the virtual-clock-dependent controller has better performances than mode-dependent and common controllers since it yields smaller ℓ_2-induced gains for the closed-loop system. The common-gain controller has the largest and most conservative design result, however, it does not need any virtual clock or mode activation detection, which is easy to be implemented in practice. Additionally, for virtual-clock-dependent controller, a larger L can lead to a better $\mathcal{H}_\infty$ performance.

VI. Conclusions

Based on the novel virtual clock conception introduced in this paper, two necessary and sufficient conditions for discrete-time switched linear systems under minimum dwell-time constraint are proposed. The lifted version is able to maintain the convex feature which plays a crucial role for some further extensions. The non-conservativeness in stability analysis can be obtained as long as the length of the virtual clock is sufficiently long. Then, taking advantage of the convexity in the lifted condition, the extensions to ℓ_2-gain computation and $\mathcal{H}_\infty$ control problems are made. It shows that the virtual clock method outperforms the mode-dependent method and common gain control method. Several numerical examples are given to illustrate the theoretical findings in this paper. This paper presents a framework of virtual clock to improve the stability analysis for switched systems under dwell time constraint. It should be mentioned that, according to Table I, the computational cost significantly increases as the number of modes and length of the cycle of virtual clock, that is the L, grow, how to reduce the computational complexity and make it applicable for switched system with large amounts of subsystems needs further studies. Moreover, how to select appropriate L to avoid unnecessary computational cost also needs further investigations in the future.

References

[1] R. DeCarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[2] D. Liberzon, *Switching in Systems and Control*. Springer Science & Business Media, 2012.

[3] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, "Stability criteria for switched and hybrid systems," *SIAM Review*, vol. 49, no. 4, pp. 545–592, 2007.

[4] M. Margaliot, "Stability analysis of switched systems using variational principles: An introduction," *Automatica*, vol. 42, no. 12, pp. 2059–2077, 2006.

[5] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: a survey of recent results," *IEEE Transactions on Automatic control*, vol. 54, no. 2, pp. 308–322, 2009.

[6] Z. Sun and S. S. Ge, *Stability Theory of Switched Dynamical Systems*. Springer Science & Business Media, 2011.

[7] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[8] A. S. Morse, "Supervisory control of families of linear set-point controllers Part I. exact matching," *IEEE Transactions on Automatic Control*, vol. 41, no. 10, pp. 1413–1431, 1996.

[9] J. P. Hespanha, "Stability of switched systems with average dwell-time," in *Proceedings of the 38th IEEE Conference on Decision and Control, 1999*, vol. 3, (Phoenix, AZ), pp. 2655–2660, IEEE, 1999.

[10] J. C. Geromel and P. Colaneri, "Stability and stabilization of continuous-time switched linear systems," *SIAM Journal on Control and Optimization*, vol. 45, no. 5, pp. 1915–1930, 2006.

[11] J. C. Geromel and P. Colaneri, "Stability and stabilization of discrete time switched systems," *International Journal of Control*, vol. 79, no. 07, pp. 719–728, 2006.

[12] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, "Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach," *International Journal of Systems Science*, vol. 32, no. 8, pp. 1055–1061, 2001.

[13] L. Zhang and H. Gao, "Asynchronously switched control of switched linear systems with average dwell time," *Automatica*, vol. 46, no. 5, pp. 953–958, 2010.

[14] L. Zhang, S. Zhuang, and P. Shi, "Non-weighted quasi-time-dependent $\mathcal{H}_\infty$ filtering for switched linear systems with persistent dwell-time," *Automatica*, vol. 54, pp. 201–209, 2015.

[15] L. Zhang and P. Shi, "Stability, ℓ_2-gain and asynchronous $\mathcal{H}_\infty$ control of discrete-time switched systems with average dwell time," *IEEE Transactions on Automatic Control*, vol. 54, no. 9, pp. 2192–2199, 2009.

[16] N. Athanasopoulos, K. Smpoukis, and R. M. Jungers, "Safety and invariance for constrained switching systems," in *Decision and Control (CDC), 2016 IEEE 55th Conference on*, pp. 6362–6367, IEEE, 2016.

[17] N. Athanasopoulos and M. Lazar, "Stability analysis of switched linear systems defined by graphs," in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pp. 5451–5456, IEEE, 2014.

[18] M. Dehghan and C.-J. Ong, "Characterization and computation of disturbance invariant sets for constrained switched linear systems with dwell time restriction," *Automatica*, vol. 48, no. 9, pp. 2175–2181, 2012.

[19] M. Dehghan and C.-J. Ong, "Computations of mode-dependent dwell times for discrete-time switching system," *Automatica*, vol. 49, no. 6, pp. 1804–1808, 2013.

[20] F. Blanchini, D. Casagrande, and S. Miani, "Modal and transition dwell time computation in switching systems: A set-theoretic approach," *Automatica*, vol. 46, no. 9, pp. 1477–1482, 2010.

[21] L. Zhang, S. Zhuang, and R. D. Braatz, "Switched model predictive control of switched linear systems: Feasibility, stability and robustness," *Automatica*, vol. 67, pp. 8–21, 2016.

[22] C. Briat and A. Seuret, "Affine characterizations of minimal and mode-dependent dwell-times for uncertain linear switched systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 5, pp. 1304–1310, 2013.

[23] L. Allerhand and U. Shaked, "Robust stability and stabilization of linear switched systems with dwell time," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 381–386, 2011.

[24] C. Briat, "Convex lifted conditions for robust ℓ_2-stability analysis and ℓ_2-stabilization of linear discrete-time switched systems with minimum dwell-time constraint," *Automatica*, vol. 50, no. 3, pp. 976–983, 2014.

[25] W. Xiang and J. Xiao, "Convex sufficient conditions on asymptotic stability and ℓ_2 gain performance for uncertain discrete-time switched linear systems," *IET Control Theory & Applications*, vol. 8, no. 3, pp. 211–218, 2014.

[26] W. Xiang, "On equivalence of two stability criteria for continuous-time switched systems with dwell time constraint," *Automatica*, vol. 54, pp. 36–40, 2015.

[27] W. Xiang, "Necessary and sufficient condition for stability of switched uncertain linear systems under dwell-time constraint," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3619–3624, 2016.

[28] W. Xiang and J. Xiao, "Stabilization of switched continuous-time systems with all modes unstable via dwell time switching," *Automatica*, vol. 50, no. 3, pp. 940–945, 2014.

[29] C. Briat, "Dwell-time stability and stabilization conditions for linear positive impulsive and switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 24, pp. 198–226, 2017.

[30] C. Briat, "Stability analysis and stabilization of stochastic linear impulsive, switched and sampled-data systems under dwell-time constraints," *Automatica*, vol. 74, pp. 279–287, 2016.

[31] P. Li, J. Lam, and K. C. Cheung, "Stability, stabilization and $\mathcal{L}_2$-gain analysis of periodic piecewise linear systems," *Automatica*, vol. 61, pp. 218–226, 2015.

[32] W. Xiang, G. Zhai, and C. Briat, "Stability analysis for LTI control systems with controller failures and its application in failure tolerant control," *IEEE Transactions on Automatic Control*, vol. 61, no. 3, pp. 811–816, 2016.

[33] G. Chesi, P. Colaneri, J. C. Geromel, R. Middleton, and R. Shorten, "A nonconservative lmi condition for stability of switched systems with guaranteed dwell time," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1297–1302, 2012.

[34] M. Philippe, R. Essick, G. E. Dullerud, and R. M. Jungers, "Stability of discrete-time switching systems with constrained switching sequences," *Automatica*, vol. 72, pp. 242–250, 2016.

[35] M. Philippe, R. Essick, G. Dullerud, and R. M. Jungers, "Extremal storage functions and minimal realizations of discrete-time linear switching systems," in *Decision and Control (CDC), 2016 IEEE 55th Conference on*, pp. 5533–5538, IEEE, 2016.

[36] Y. Wang, N. Roohi, G. E. Dullerud, and M. Viswanathan, "Stability analysis of switched linear systems defined by regular languages," *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2568–2575, 2017.

[37] R. Essick, J.-W. Lee, and G. E. Dullerud, "Control of linear switched systems with receding horizon modal information," *IEEE Transactions on Automatic Control*, vol. 59, no. 9, pp. 2340–2352, 2014.

[38] H. K. Khalil, *Nonlinear Systems*. New Jersey: Prentice Hall, 2003.